

Ex: Duel

Two persons, A and B, engage in a duel with (water)-guns.

The persons fire at each others, taking turns. A starts.

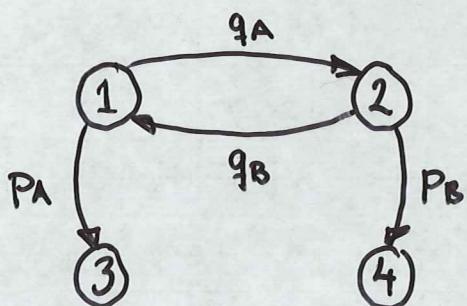
The probability that A hits B is P_A .

— “ — B hits A is P_B

The duel ends when one of them is hit.

This can be modelled as a Markov chain with the states

$$X_t = \begin{cases} 1 & \text{it is A's turn to shoot} \\ 2 & -\text{--} B:\text{s} \text{ --} \text{--} \\ 3 & \text{A has shot B} \\ 4 & B - \text{--} A. \end{cases}$$



Where $q_A = 1 - P_A$
 $q_B = 1 - P_B$.

The (one-step) transition matrix is

$$P = \begin{bmatrix} 0 & q_A & P_A & 0 \\ q_B & 0 & 0 & P_B \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

States 3 and 4 are absorbing

States 1 and 2 are transient

\Rightarrow Not irreducible.

Let $P_A = \frac{1}{3}$, $P_B = \frac{1}{2}$

$$\Rightarrow P = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solve

$$\left\{ \begin{array}{l} \pi = \pi P \\ \sum \pi_i = 1 \end{array} \right.$$

\Leftrightarrow

$$\left\{ \begin{array}{l} \pi_1 = \frac{1}{2} \pi_2 \\ \pi_2 = \frac{2}{3} \pi_1 \\ \pi_3 = \frac{1}{3} \pi_1 + \pi_3 \\ \pi_4 = \frac{1}{2} \pi_2 + \pi_4 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{array} \right. \Rightarrow \begin{array}{l} \pi_1 = 0 \\ \pi_2 = 0 \\ \pi_3 = 1 \\ \pi_4 = 1 \end{array}$$

No unique solution!

Let $p(t) = (p_1(t) \ p_2(t) \ p_3(t) \ p_4(t))$ $p_i(t) = \Pr(\text{in state } i \text{ at time } t)$

$$p(t) = p(0) P^t$$

$$p(0) = (1 \ 0 \ 0 \ 0) \quad \text{A starts shooting.}$$

$$p(1) = p(0) P = (0 \ \frac{2}{3} \ \frac{1}{3} \ 0)$$

$$p(2) = p(0) P^2 = (\frac{1}{3} \ 0 \ \frac{1}{3} \ \frac{1}{3})$$

$$p(3) = p(0) P^3 = (0 \ \frac{2}{9} \ \frac{4}{9} \ \frac{3}{9})$$

\downarrow

$$p(\infty) = (0 \ 0 \ \frac{1}{2} \ \frac{1}{2})$$

fair game!