Example on continuous time Markov chains

Consider a simple example with only two states:
"Functioning" (=state 0) and "Failed" (=state 1).
Let $X(t)$ = the state of the system at time t.

Assumptions:

$$P\left(X(t+h)=1 \mid X(t)=0\right)=\lambda h+o(h),$$

$$P\left(X(t+h)=0 \mid X(t)=0\right)=1-\lambda h+o(h),$$

$$P\left(X(t+h)=0 \mid X(t)=1\right)=\mu h+o(h),$$

$$P\left(X(t+h)=1 \mid X(t)=1\right)=1-\mu h+o(h),$$

Interpretation:
When the system is functioning, the time until it fails is exponentially distributed with intensity λ.
When the system is failed, the time until it will function is exponentially distributed with intensity μ.
The expected functioning time between failures is $1/\lambda$.
The expected repair time is $1/\mu$.
λ is the *failure rate*, while μ is the *repair rate*.

The matrix $\mathbf{Q}$ for this example is

$$\mathbf{Q}=\begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}.$$

Let $p_0(t)=P(X(t)=0)$, $p_1(t)=P(X(t)=1)$ and $\mathbf{p}(t)=[p_0(t)\ p_1(t)]$.
Then the differential equation system $\dot{\mathbf{p}}(t)=\mathbf{p}(t)\mathbf{Q}$ becomes

$$\dot{p}_0(t)=-\lambda p_0(t)+\mu p_1(t),$$

$$\dot{p}_1(t)=\lambda p_0(t)-\mu p_1(t).$$

The solution of this system is determined by the matrix exponential $\mathbf{p}(t)=\mathbf{p}(0)\exp\{\mathbf{Q}t\}$, which gives

$$p_0(t)=\frac{\mu}{\lambda+\mu}+\left(p_0(0)-\frac{\mu}{\lambda+\mu}\right)e^{-(\lambda+\mu)t},$$

$$p_1(t)=\frac{\lambda}{\lambda+\mu}+\left(p_1(0)-\frac{\lambda}{\lambda+\mu}\right)e^{-(\lambda+\mu)t},$$

from which it follows that (exponentially fast convergence)

$$p_0(t) \to \frac{\mu}{\lambda+\mu} = \pi_0, \text{ and } p_1(t) \to \frac{\lambda}{\lambda+\mu} = \pi_1, \text{ when } t \to \infty.$$

Note that this asymptotic distribution $\pi = (\pi_0,\ \pi_1)$ is the unique solution to the steady-state system of linear equations

$$\pi\mathbf{Q} = \mathbf{0}, \qquad \pi_0 + \pi_1 = 1.$$

i.e.

$$\begin{pmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{pmatrix} \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} = \mathbf{0}.$$

and

$$\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} = 1.$$

(That can be solved for π without resorting to matrix exponentials)