

Formula-sheet at the exam in SF2863 and SF2862, June 2010

If events occur according to a Poisson process with rate λ , where τ denotes the time between two consecutive events, and $X(T)$ denotes the number of events on the time interval $[0, T]$, then

$$P(\tau \leq t) = 1 - e^{-\lambda t}, \quad P(X(T) = \ell) = \frac{(\lambda T)^\ell}{\ell!} e^{-\lambda T}, \quad E[\tau] = 1/\lambda, \quad E[X(T)] = \lambda T.$$

Markov chain in discrete time.

\mathbf{P} = the matrix with elements $p_{ij} = P(X_{n+1} = j \mid X_n = i)$.

$\mathbf{p}^{(n)}$ = the row vector with components $p_j^{(n)} = P(X_n = j)$. Then $\mathbf{p}^{(n+1)} = \mathbf{p}^{(n)}\mathbf{P}$.

The row vector π defines a stationary distribution if $\pi = \pi\mathbf{P}$, $\sum_j \pi_j = 1$ and $\pi_j \geq 0$.

Markov chain in continuous time (also called Markov process with discrete state space).

$\mathbf{P}(h)$ the matrix with elements $p_{ij}(h) = P(X(t+h) = j \mid X(t) = i)$.

$\mathbf{p}(t)$ = the row vector with components $p_j(t) = P(X(t) = j)$. Then $\mathbf{p}(t+h) = \mathbf{p}(t)\mathbf{P}(h)$.

Assumption: $p_{ij}(h) = q_{ij}h + o(h)$ if $j \neq i$, while

$$p_{ii}(h) = 1 + q_{ii}h + o(h) = 1 - q_i h + o(h), \quad \text{where } q_i = -q_{ii} = \sum_{j \neq i} q_{ij}.$$

Thus, $\mathbf{P}(h) \approx \mathbf{I} + h\mathbf{Q}$ and $(\mathbf{p}(t+h) - \mathbf{p}(t))/h \approx \mathbf{p}(t)\mathbf{Q}$ for small $h > 0$.

The row vector π defines a stationary distribution if $\pi\mathbf{Q} = \mathbf{0}$, $\sum_j \pi_j = 1$ and $\pi_j \geq 0$.

The system $\pi\mathbf{Q} = \mathbf{0}$ can be written $\sum_{i \neq j} \pi_i q_{ij} + \pi_j q_{jj} = 0$, for all j , or

$\pi_j \sum_{k \neq j} q_{jk} = \sum_{i \neq j} \pi_i q_{ij}$ ("jumps out from state j = jumps into state j ").

Some quantities and relations in queueing theory (where P_n corresponds to π_n above):

$$L = \sum_{n=0}^{\infty} nP_n, \quad L_q = \sum_{n=s}^{\infty} (n-s)P_n, \quad \bar{\lambda} = \sum_{n=0}^{\infty} \lambda_n P_n, \quad L = \bar{\lambda}W, \quad L_q = \bar{\lambda}W_q$$

$$M/M/1: \quad \rho = \lambda/\mu < 1, \quad P_0 = 1 - \rho, \quad P_n = \rho^n P_0, \quad L = \frac{\rho}{1 - \rho}.$$

$$M/M/2: \quad \lambda_n = \lambda \text{ for } n \geq 0, \quad \mu_1 = \mu, \quad \mu_n = 2\mu \text{ for } n \geq 2, \quad \rho = \lambda/(2\mu) < 1, \\ P_0 = \frac{1 - \rho}{1 + \rho}, \quad P_n = 2\rho^n P_0 \text{ for } n \geq 1, \quad L = \frac{2\rho}{1 - \rho^2}.$$

Jackson queueing networks.

Calculate $\lambda_1, \dots, \lambda_m$ from $\lambda_j = a_j + \sum_i \lambda_i p_{ij}$. Check $\lambda_j < s_j \mu_j$.

Analyze each service facility to obtain $P(N_j = n_j)$.

Then $P(N_1 = n_1, \dots, N_m = n_m) = \prod_j P(N_j = n_j)$.

W_1, \dots, W_m can be obtained from $W_i = V_i + \sum_j p_{ij} W_j$, where $V_i = L_i / \lambda_i$.

Some deterministic inventory models.

EOQ with shortage not permitted: Minimize $\frac{Kd}{Q} + cd + \frac{hQ}{2}$.

$$C_i = \min_j \{C_i^{(j)} \mid i \leq j \leq n\}, \quad \text{where } C_i^{(j)} = C_{j+1} + K + h(r_{i+1} + 2r_{i+2} + \dots + (j-i)r_j).$$

Some stochastic inventory models.

$$C(S) = cS + pE[(\xi - S)^+] + hE[(S - \xi)^+].$$

If ξ is a continuous non-negative random variable then

$$E[(\xi - S)^+] = \int_S^\infty (t - S)f_\xi(t)dt, \quad E[(S - \xi)^+] = \int_0^S (S - t)f_\xi(t)dt,$$

and $C'(S) = c + p(F_\xi(S) - 1) + hF_\xi(S)$.

If ξ is a non-negative integer-valued random variable then S is integer and

$$E[(\xi - S)^+] = \sum_{j=S}^\infty (j - S)p_\xi(j), \quad E[(S - \xi)^+] = \sum_{j=0}^S (S - j)p_\xi(j),$$

and $C(S + 1) - C(S) = c + p(F_\xi(S) - 1) + hF_\xi(S)$.

Marginal allocation for generating efficient solutions to the pair (f, g) , where f and g are integer-convex separable functions, f decreasing and g increasing in the non-negative integer variables x_1, \dots, x_n .

Generate a table in which the j :th column contains the quotients

$$-\Delta f_j(0)/\Delta g_j(0), -\Delta f_j(1)/\Delta g_j(1), -\Delta f_j(2)/\Delta g_j(2), \dots$$

Let all the quotients in the table be uncanceled.

Initiate the variables to their smallest feasible values and repeat the following:

Let ℓ be the number of the column with the largest uncanceled quotient.

Cancel this quotient, and increase the ℓ :th variable x_ℓ by one.

Finite horizon MDP recursion (discounting if $0 < \alpha < 1$, no discounting if $\alpha = 1$):

$$V_i^{(n)} = \min_k \{ C_{ik} + \alpha \sum_j p_{ij}(k) V_j^{(n-1)} \} \quad (\text{backward time}).$$

LP formulation for MDP **without** discounting:

$$\begin{aligned} & \text{minimize} && \sum_i \sum_k C_{ik} y_{ik} \\ & \text{subject to} && \sum_i \sum_k y_{ik} = 1, \\ & && \sum_k y_{jk} - \sum_i \sum_k p_{ij}(k) y_{ik} = 0, \text{ for all } j, \\ & && y_{ik} \geq 0, \text{ for all } i \text{ and } k. \end{aligned}$$

Policy improvement algorithm for MDP **without** discounting:

1. For a given policy, calculate v_0, \dots, v_M and g from $v_M = 0$ and $g + v_i = C_{i,d_i} + \sum_j p_{ij}(d_i) v_j$.
2. The current policy is optimal if $g + v_i = \min_k \{ C_{ik} + \sum_j p_{ij}(k) v_j \}$. Otherwise, define a new policy by letting $d_i =$ a minimizing k above. Then go to 1.

LP formulation for MDP **with** discounting:

$$\begin{aligned} & \text{minimize} && \sum_i \sum_k C_{ik} y_{ik} \\ & \text{subject to} && \sum_k y_{jk} - \alpha \sum_i \sum_k p_{ij}(k) y_{ik} = \beta_j, \text{ for all } j, \\ & && y_{ik} \geq 0, \text{ for all } i \text{ and } k. \end{aligned}$$

where the constants in the right hand sides should satisfy $\beta_j > 0$ and $\sum_j \beta_j = 1$.

Policy improvement algorithm for MDP **with** discounting:

1. For a given policy, calculate V_0, \dots, V_M from $V_i = C_{i,d_i} + \alpha \sum_j p_{ij}(d_i) V_j$.
2. The current policy is optimal if $V_i = \min_k \{ C_{ik} + \alpha \sum_j p_{ij}(k) V_j \}$. Otherwise, define a new policy by letting $d_i =$ a minimizing k above. Then go to 1.

Note: No calculator at the exam!