

## Home assignment number 1, 2013, in SF2863 Systems Engineering.

*Instructor:* Per Enqvist, [penqvist@math.kth.se](mailto:penqvist@math.kth.se).

In this home assignment it is allowed to cooperate in groups of at most four persons each. Limited cooperation with other groups is also allowed, but should be referred to in the report.

One report per group is to be handed in, and you should describe in your own words how the problem was solved. Make the description rich enough so that it is totally clear how you defined your states and modelled the problem and describe how you implemented the simulations. It is not allowed to copy parts of another groups report or computer code! The answers to the questions in the assignment should be given in the main report. (Not in your computer printouts.) Relevant print-outs and plots should be included in the report. You should also write down how much time each of you have worked on the different parts. Computer code should be held available on request.

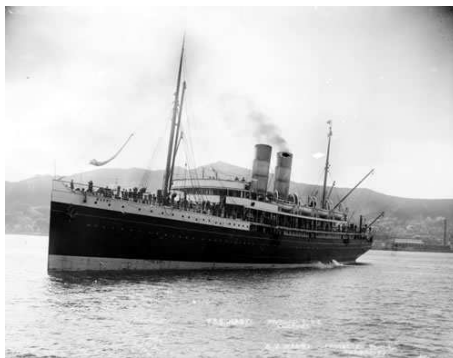
State your name, personal number and email adress on the front of the report.

A written report (printed out on paper) should be handed in to the instructor before 13.15, Tuesday, November 20. Note that this deadline is sharp. (Note that the assignment is completely voluntary, but that a correct report that is handed in in time will grant you bonus points at the final exam.)

This home assignment can grant a maximum of two bonus points on the exam.

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In this home assignment a simple Markov process in continuous time will be examined. We consider a big (old) ferry with two engines that each drives one propeller. If one engine breaks down the ferry can still move, but it runs a bit slower, and if both breaks down the ferry stops.



Assume that engine number 1 breaks down after some time that is modelled as a stochastic variable with  $\text{Exp}(\lambda_1)$  distribution. Similarly, engine number 2 breaks down after some time that is  $\text{Exp}(\lambda_2)$  distributed. Three repairmen are working on the ferry for repairing the engines. The repair time is  $\text{Exp}(n_k\mu)$  distributed, where  $n_k$  is the number of repairmen working on engine  $k$ .

Let  $V(t)$  denote the speed of the ferry at time  $t$ . If both the engines are working the ferry is running at a speed which is  $v$ , if only engine 1 is working the speed is  $v_1$  and if only engine 2 is working the speed is  $v_2$ .

Now, the system is a Markov process that will depend on how many workers are assigned to repair the engines if they break down. If only one engine is broken it is obvious that

it is optimal to let all three repairmen work on it. If both engines are broken it is not so clear how to distribute the workers. To decide this you should for each different repair strategy determine the average speed of the ferry at steady state and this way choose the best strategy.

You should do this in two different ways.

First, all of you should determine the average speed using the theory of Markov processes. For this approach you need to define the states of the Markov process, the intensity matrix, motivate why there should exist a stationary distribution and determine it. The average speed can then be determined from this stationary distribution.

Second you should simulate the system numerically and determine the average speed by ergodic estimates.

This will be done in two different ways; half of the members in the group will use a time discretization approach, while the second half will directly simulate the continuous time process.

### Discretization approach

Now a discretization of the continuous time process will be made.

Start the process at time 0 with both engines of the ferry functioning. Then make a discretization of the time axis so that we only consider the process at times  $t_k = kh$  for  $k = 0, 1, 2, \dots, N$ . Then the probability of a jump to another state during the time interval  $[t_k, t_{k+1}]$  can be determined (approximatively) using the intensity matrix for the continuous time Markov process. (Without explicitly using the exponential distributions.) If the time step  $h$  is small this approximation is good. These probabilities form the transition matrix of a discrete time Markov chain that will approximate the continuous time one. Check if the transition matrix you obtain has row sums equal to one.

Now use this discretization to simulate the process. If you use Matlab, a stochastic variable with uniform probability in  $[0, 1]$  can be generated using the command `rand`. Then take averages to determine estimates of the expected speed of the ferry.

*The approach just described has also the purpose of showing the connection between discrete time and continuous time Markov chains.*

### Continuous time approach

It is of course also possible to simulate the continuous time process. Start the process at time 0 with both engines of the ferry functioning. Then you should use that the jump times are exponentially distributed. If you use Matlab an exponentially distributed variable can be generated using the command `exprnd`, where you specify the mean of the variable (which is inversely proportional to the intensity). Then you use the exponential distribution to determine how long you remain in a state. You can use several exponential distributions, or one exponential distribution and a uniform probability generated using the command `rand`, to determine how long you remain in a state and which will be the next state. Run the simulation for a time interval  $[0, T]$ .

### Parameter values

- $v = 18$ ,  $v_1 = 10$ , and  $v_2 = 14$ .
- $\lambda_1 = 10$ ,  $\lambda_2 = 20$ , and  $\mu = 8$
- $h = 0.001$
- Choose  $N$  and  $T$  large enough so that the ergodic estimates are within 1% of the theoretical means.

What is the computation time for this choice ?

(use `tic`, `toc` or `profile` to determine this)

### Hints

Since a large number of time steps have to be made to get reasonable ergodic estimates you might run into memory problems if you store all the old states in the simulations. Please note that it is enough to keep track on how long time the Markov chain has spent so far in each state to determine the estimates for the stationary probabilities and the average speed.

Now that you have performed the simulations you should meet together and explain to each other how you performed the simulations. Compare the results of the two different simulation approaches, determine drawbacks and advantages of the two approaches and write down your conclusions in the report.

### Theory question

A last exercise is given in the box below:

If  $\tau$  is the stochastic variable that describes the time to the next jump of the continuous time Markov process  $X(t)$  with intensity  $q$ , then we know that  $\tau$  must be  $\text{Exp}(q)$  distributed. That means that  $\tau$  has the distribution function  $F_\tau(t) = P(\tau \leq t) = 1 - e^{-qt}$ .

**Show that**

if  $\tau$  has the  $\text{Exp}(q)$  distribution, then  $P(\tau \geq 1) = P(\tau \geq 1.5 \mid \tau \geq 0.5)$ .

This is a property any time-homogeneous Markov process should have, why ?

**Good luck!**