KTH Mathematics

Suggested solutions for the exam in SF2863 Systems Engineering.
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1. We can think of the support center as a Jackson network. The reception is a $M|M|2$ queue with arrival intensity λ_R and service intensity $\mu_R = 16$ per server. Frasse's service is a $M|M|1$ queue with arrival intensity λ_F and service intensity $\mu_F = 10$ per server. The reception is a $M|M|1$ queue with arrival intensity λ_H and service intensity $\mu_R = 16$ per server.

(a) Let $\lambda_0 = 16$ be the arrival intensity from the outside and λ_R be the intensity in to the reception. The traffic balance equations are $\lambda_0 + 1/4\lambda_F + 1/\lambda_H = \lambda_R$, $\lambda_F = 1/3\lambda_R$ and $\lambda_H = 1/2\lambda_R$ which yields, $\lambda_R = 24$, $\lambda_F = 8$ and $\lambda_H = 12$. We can now check that the low traffic requirements are satisfied, i.e., that $\lambda_R = 24 < 2 * \mu_A = 32$, $\lambda_F = 8 < \mu_F = 10$, and $\lambda_H = 12 < \mu_H = 16$.

The probability that a random customer gets the problem solved is equal to the ratio of callers from the outside divided by callers being helped $= (6+6)/16 = 3/4$.

(b) Let V_R, V_F and V_H be the average time it takes from a call arrives to one of the service stations until it leaves it, i.e., $V_R = L_R/\lambda_R = 2\rho_R/(1-\rho_R^2)/\lambda_R = 24/7/24 = 1/7$, $V_F = L_F/\lambda_F = \rho_F/(1-\rho_F)/\lambda_F = 4/8 = 1/2$, and $V_H = L_H/\lambda_H = \rho_H/(1-\rho_H)/\lambda_H = 3/12 = 1/4$.

Letting W_R be the average time from a call arrives to station R until it exits the system , W_F be the average time from a call arrives to station F until it exits the system, and W_H be the average time from the call arrives to station H until it exits the system, then

$$W_R = V_R + 1/3W_F + 1/2W_H$$

$$W_F = V_F + 1/4W_R$$

$$W_H = V_H + 1/2W_R$$

and $W_R = 73/112$.

2. (a) Introduce the state, $s_\ell =$ how many dollars Frasse has day ℓ. We know that $s_0 = 1000$, and $s_\ell \in \{0, 500, 1000, 1500, 2000\}$, where all states are not reachable at all times. The state 0 is absorbing.

Let x_ℓ be the strategy of Frasse day ℓ, and let it be 1 if he plays fair and 0 if he counts cards.

Define the value function $V_\ell^*(s)$ to be the maximal expected winning of Frasse if he starts day ℓ with s dollars and use the optimal playing strategy.

The DynP equation becomes

$$V_\ell^*(s) = \max_{x=0,1} \{\mathrm{E}\ V_\ell^*(s,x)\} =$$

$$= \max\left\{0.6V_{\ell+1}^*(s+500) + 0.4V_{\ell+1}^*(s-500), 0.7V_{\ell+1}^*(s+500) + 0.1V_{\ell+1}^*(s-500),\right\}$$

The boundary condition is that $V_2^*(s) = s$.

(b) Use the recursion.
First $V_2(s) = s$.
Then $V_1^*(1500) = \max\{0.6V_2^*(2000) + 0.4V_2^*(1000), 0.7V_2^*(2000) + 0.1V_2^*(1000)\} = 1600$ for $x_1 = 1$,
$V_1^*(500) = \max\{0.6V_2^*(1000) + 0.4V_2^*(0), 0.7V_2^*(1000) + 0.1V_2^*(0)\} = 700$ for $x_1 = 0$,
$V_1^*(0) = 0$

Then $V_0^*(1000) = \max\{0.6V_1^*(1500) + 0.4V_2^*(500), 0.7V_1^*(1500) + 0.1V_1^*(500)\} = 1100$ for $x_1 = 0, 1$.

The first day the strategy of Frasse does not matter, both are optimal. The second day he should play fair if he won the first day and he should count cards if he lost the first day, and if got thrown out the first day he can do nothing.

3. (a) Let f be equal to $-p$, and $g = n_H + n_L + n_C$ be the total number of boxes. (Maximizing p is equivalent to minimizing f) Then both f and g are clearly separable, so we can just check the properties of f_k and g_k for $k = 1, 2, 3$.
Note that $\Delta f_k(x) = -\Delta p_k$

n	$\Delta p_H(n)$	$\Delta p_L(n)$	$\delta p_C(n)$
0	5	3	7
1	3	2	5
2	2	1	3
3	1	1	1

so it is decreasing and $\Delta g_k(x) = 1$ so it is increasing.
Note that $\Delta^2 f_k(x) = -\Delta^2 p_k$

n	$\Delta^2 p_H(n)$	$\Delta^2 p_L(n)$	$\Delta^2 p_C(n)$
0	-2	-1	-2
1	-1	-1	-2
2	-1	0	-2

and $\Delta^2 g_k(x) = 0$ they are both integer convex for the given numbers.

(b) When we apply the marginal allocation algorithm we want to compare the quotients $-\Delta f_k(x)/\Delta g_k(x)$ and find the largest elements when $k = 1, 2, 3$ and $n = 1, 1, 2, \cdots$.

n	$\Delta f_H(n)$	$\Delta f_L(n)$	$\Delta f_C(n)$
0	5	3	7
1	3	2	5
2	2	1	3
3	1	1	1

The largest element is 7, so $n^{(1)} = (0, 0, 1)$ is the optimal allocation for sum of n is 1 and $f(n^{(1)}) = -7$, $g(n^{(1)}) = 1$.

The largest element is 5 (in two places), so $n^{(2)} = (1, 0, 1)$ is an optimal allocation for sum of n is 2 and $f(n^{(2)}) = -12$, $g(n^{(2)}) = 2$.

The largest element is 5, so $n^{(3)} = (1, 0, 2)$ is the optimal allocation forsum of n is 3 and $f(n^{(3)}) = -17$, $g(n^{(3)}) = 3$

The largest element is 3 (in three places), so $n^{(4)} = (1, 1, 2)$ is an optimal allocation for sum of n is 4 and $f(n^{(4)}) = -20$, $g(n^{(4)}) = 4$

The largest element is 3 (in two places), so $n^{(5)} = (2, 1, 2)$ is an optimal allocation for sum of n is 5 and $f(n^{(5)}) = -23$, $g(n^{(5)}) = 5$

The largest element is 3, so $n^{(6)} = (2, 1, 3)$ is the optimal allocation for sum of n is 6 and $f(n^{(6)}) = -26$, $g(n^{(6)}) = 6$

The largest element is 2 (in two places), so $n^{(7)} = (3, 1, 3)$ is an optimal allocation for sum of n is 7 and $f(n^{(7)}) = -28$, $g(n^{(7)}) = 7$

The largest element is 2 , so $n^{(8)} = (2, 1, 2)$ is the optimal allocation for sum of n is 8 and $f(n^{(8)}) = -30$, $g(n^{(8)}) = 8$

4. (a) We need to keep track of the mood of Heathcliff, define the state

$$s_k = \begin{cases} 1 & \text{if Heathcliffs mood is up at beginning of day } k \\ 0 & \text{if Heathcliffs mood is down at beginning of day } k \end{cases}$$

Define the decisions

$$x_k = \begin{cases} 1 & \text{if Frasse does nothing day } k \\ 2 & \text{if Frasse buys a medal day } k \\ 3 & \text{if Frasse takes Heathcliff out on day (evening) } k \end{cases}$$

In the costs we include the revenue with negative sign. Then the costs of making decision $x_k = 1$ is $C_{11} = -60$ if $s_k = 1$ and $C_{01} = -30$ if $s_k = 0$, the cost of making decision $x_k = 2$ is $C_{12} = -55$ if $s_k = 1$ and $C_{02} = -25$ if $s_k = 0$, and the cost of making decision $x_k = 3$ is $C_{13} = -30$ if $s_k = 1$ and $C_{03} = 0$ if $s_k = 0$.
Starting policy:
Always make decision $x_k = 1$.
Use the policy iteration algorithm. Let $v_0 = 0$, then the value determination equations

$$g + v_1 = -60 + 0.8v_1 + 0.2v_0$$

$$g + v_0 = -30 + 0.4v_1 + 0.6v_0$$

gives $g = -50$, $v_1 = -50$.
To find out if it is optimal we do one step of the policy iteration.
For $i = 1$ (Heathcliff is up)

$$\min_{k=1,2,3}\{C_{1k} + (p_{11}(k)v_1 + p_{10}(k)v_0)\} =$$

$$= \min\{C_{11}+(p_{11}(1)v_1+p_{10}(1)v_0), C_{12}+(p_{11}(2)v_1+p_{10}(2)v_0), C_{13}+(p_{11}(3)v_1+p_{10}(3)v_0),\}$$
$$= \min\{\underbrace{-60 + (0.8v_1 + 0.2v_0)}_{-100}, \underbrace{-55 + (0.9v_1 + 0v_0)}_{-100}, \underbrace{-30 + (v_1)}_{-80},\} = -100 = g+v_1 \text{ for } k = 1.\text{(and } k = 2)$$

For $i = 0$ (Heathcliff is down)

$$\min_{k=1,2,3}\{C_{0k} + (p_{00}(k)v_0 + p_{01}(k)v_1)\} =$$

$$= \min\{C_{01}+(p_{01}(1)v_1+p_{00}(1)v_0), C_{02}+(p_{01}(2)v_1+p_{00}(2)v_0), C_{03}+(p_{01}(2)v_1+p_{00}(2)v_0),\}$$
$$= \min\{\underbrace{-30 + (0.4v_1 + 0.6v_0)}_{-50}, \underbrace{-25 + (0.6v_1 + 0.4v_0)}_{-55}, \underbrace{0 + (v_1)}_{-50},\} = -55 \text{ for } k = 2.$$

So the policy $R_1 = [2\ 1]$, to buy a medal if Heathcliff is down and do nothing when he is up is better than $R_0 = [1\ 1]$, i.e. to always do nothing.
Let $v_0 = 0$, then solve the value determination equations again for the new policy

$$g + v_1 = -60 + 0.8v_1 + 0.2v_0$$
$$g + v_0 = -25 + 0.6v_1 + 0.4v_0$$

gives $g = -51.25$, $v_1 = -43.75$.
The expected revenue increases by 1.25 dollars per day.

(b) Let

$$y = \begin{bmatrix} y_{01} & y_{02} & y_{03} & y_{11} & y_{12} & y_{13} \end{bmatrix}^T$$

Then

$$c = \begin{bmatrix} C_{01} & C_{02} & C_{03} & C_{11} & C_{12} & C_{13} \end{bmatrix}^T = \begin{bmatrix} -30 & -25 & 0 & -60 & -55 & -30 \end{bmatrix}^T,$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1-p_{00}(1) & 1-p_{00}(2) & 1-p_{00}(3) & -p_{10}(1) & -p_{10}(2) & -p_{10}(3) \\ -p_{01}(1) & -p_{01}(2) & -p_{01}(3) & 1-p_{11}(1) & 1-p_{11}(2) & 1-p_{11}(3) \end{bmatrix} =$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1-0.6 & 1-0.4 & 1-0 & -0.2 & -0.1 & 0 \\ -0.4 & -0.6 & -1 & 1-0.8 & 1-0.9 & 1-1 \end{bmatrix}$$

and $b = [1\ 0\ 0]^T$.
Then solving the LP gives

$$y = \begin{bmatrix} 0 & 1/4 & 0 & 0 & 3/4 & 0 \end{bmatrix}^T$$

then $\pi_0 = y_{01} + y_{02} + y_{03} = 1/4$ and $\pi_0 = y_{01} + y_{02} + y_{03} = 3/4$.
Finally $D_{ik} = y_{ik}/\pi_i$, so $D_{01} = 0$, $D_{02} = 1$, $D_{03} = 0$, $D_{11} = 0$, $D_{12} = 1$, $D_{13} = 0$, determines the optimal deterministic policy which is to always buy Heathcliff an employee of the day medal.

5. This can be solved using the newsboy problem formulation. Let $p = 2$, $h = 8$ and $c = -5$.

Then the optimal target price satisfies $\frac{p-c}{p+h} = \frac{7}{10} = F(S^*)$.

F is the cumulative distribution function of the stock prize, which is assumed to be uniform on the interval (0,1000), i.e. $F(t) = t/1000$ on the interval. $F(700) = 0.7$ and therefore $S^* = 700$.

The optimal expected profit is given by

$$C(700) = -5 \cdot 700 + 2 \int_{700}^{1000} \frac{t - 700}{1000} dt + 8 \int_{0}^{700} \frac{700 - t}{1000} dt = -1550.$$