KTH Mathematics

Suggested solutions for the exam in SF2863 Systems Engineering.
June 12, 2012 14.00–19.00

Examiner: Per Enqvist, phone: 790 62 98

---

1. We can think of the farm as a Jackson network. The strawberry field is modelled as a $M|M|\infty$ queue with arrival intensity λ_S and service intensity $\mu_S^\infty = 60/40 = 1.5$, followed by a $M|M|1$ queue with the same arrival intensity λ_S and with service intensity $\mu_S^1 = 100$. The apple garden is modelled as a $M|M|\infty$ queue with arrival intensity λ_A and service intensity $\mu_A^\infty = 60/20 = 3$, followed by a $M|M|1$ queue with the same arrival intensity λ_A and with service intensity $\mu_A^1 = 120$.

(a) Let $\lambda_0 = 100$ be the arrival intensity from the outside. The traffic balance equations are $p\lambda_0 + 0.2\lambda_A = \lambda_S$, $(1-p)\lambda_0 + 0.5\lambda_S = \lambda_R$, which yields, $\lambda_S = (80p+20)/0.9$, and $\lambda_A = 100 + 10/0.9 - (100 - 40/0.9)p$. We can now check that the low traffic requirements are satisfied, i.e., that $\lambda_S < \mu_S^1 = 100$ and $\lambda_A < \mu_A^1 = 120$. The first equation tells us that $p < 7/8$ must hold in order to obtain steady state.

(b) With $p = 0.2$ we get $\lambda_A = 100$ and $\lambda_S = 40$.
The probability of no customers in the payment offices is $(1-\rho_S)(1-\rho_A) = (1 - 40/100)(1 - 100/120) = 0.1$ so it is 10%.

(c) The average number of strawberry pickers is $L_S = \lambda_S/\mu_S^\infty = 40/1.5$ and the average number of apple pickers is $L_A = \lambda_A/\mu_A^\infty = 100/3$.

(d) We have to add the average number of people in the two payment offices, which are $L_S^1 = \rho_S/(1-\rho_S) = 0.4/0.6 = 2/3$ and $L_A^1 = \rho_A/(1-\rho_A) = (5/6)/(1/6) = 5$. So the total number is $80/3 + 100/3 + 2/3 + 5 = 197/3$.

(e) Let V_R, V_F and V_H be the average time it takes from a call arrives to one of the service stations until it leaves it, i.e., $V_R = L_R/\lambda_R = 2\rho_R/(1-\rho_R^2)/\lambda_R = 24/7/24 = 1/7$, $V_F = L_F/\lambda_F = \rho_F/(1-\rho_F)/\lambda_F = 4/8 = 1/2$, and $V_H = L_H/\lambda_H = \rho_H/(1-\rho_H)/\lambda_H = 3/12 = 1/4$.
Letting W_S be the average time from a customer arrives to the strawberry fields until it exits the system , W_A be the average time from a call arrives to the apple gardens until it exits the system, then

$$W_S = V_S + V_S^1 + 1/2 W_A$$

$$W_A = V_A + V_A^1 + 1/5 W_S$$

where $V_S = L_S/\lambda_S$, $V_S^1 = L_S^1/\lambda_S$, $V_A = L_A/\lambda_A$, $V_A^1 = L_A^1/\lambda_A$ are the average times for passing once through the system.
Then $W_A = 178/135$ and $W_S = 145/108$.
The average time in the system for a random arriving customer is then

$$0.2W_s + 0.8W_A = 397/300.$$

2. This is a deterministic periodic-review inventory model. Let
$n =$ *the number of considered weeks*
$r_i =$ *the demand at week* i
Here $n = 4$ and $r_1 = r_2 = r_3 = r_4 = 100$.

The total cost consists of three parts: The ordering costs for orders, the holding costs and the fertilizer cost. The latter is $1000 \cdot (r_1 + r_2 + r_3 + r_4) = 400000$ for all feasible order plans, so this unavoidable cost may be ignored when searching for an optimal order plan.

Let $C_i^{(j)}$ = *the minimal remaining (ordering + holding) costs from week* i*, given that the inventory is empty at the end of week* $i-1$ *and then filled in such a way that the next time it will be empty is by the end of week* j

Then $C_i^{(j)} = K + h(r_{i+1} + 2r_{i+2} + \cdots + (j-i)r_j) + C_{j+1}$.

Further, let
C_i = *the minimal remaining (ordering+holding) cost from week* i*, given that the inventory is empty at the end of week* $i-1$.

Then $C_i = \min\{C_i^{(i)}, C_i^{(i+1)}, \cdots, C_i^{(n)}\}$.

(a) Here, $K = 700$ and $h = 3$. We then get that
$C_4 = C_4^{(4)} = 700$

$C_3^{(4)} = 700 + 300 = 1000$
$C_3^{(3)} = 700 + C_4 = 1400$
$C_3 = \min\{C_3^{(3)}, C_3^{(4)}\} = 1000$

$C_2^{(4)} = 700 + 300 + 600 = 1600$
$C_2^{(3)} = 700 + 300 + C_4 = 1700$
$C_2^{(2)} = 700 + C_3 = 1700$
$C_2 = \min\{C_2^{(2)}, C_2^{(3)}, C_2^{(4)}\} = 1600$

$C_1^{(4)} = 700 + 300 + 600 + 900 = 2500$
$C_1^{(3)} = 700 + 300 + 600 + C_4 = 2300$
$C_1^{(2)} = 700 + 300 + C_3 = 2000$
$C_1^{(1)} = 700 + C_2 = 2300$
$C_1 = \min\{C_1^{(1)}, C_1^{(2)}, C_1^{(3)}, C_1^{(4)}\} = 2000$

The optimal plan is to order 200 kilo before the first week and 200 kilo before the third week.

(b) Assume that the cost is now $3 + c$.
$C_4 = C_4^{(4)} = 700$

$C_3^{(4)} = 700 + 300 + 100c = 1000 + 100c$
$C_3^{(3)} = 700 + C_4 = 1400$
$C_3 = \min\{C_3^{(3)}, C_3^{(4)}\} = 1000 + 100c$
as long as $c \leq 4$.

$C_2^{(4)} = 700 + 300 + 600 + 300c = 1600 + 300c$
$C_2^{(3)} = 700 + 300 + 100c + C_4 = 1700 + 100c$
$C_2^{(2)} = 700 + C_3 = 1700 + 100c$
$C_2 = \min\{C_2^{(2)}, C_2^{(3)}, C_2^{(4)}\} = 1600 + 300c$
as long as $c \leq 1/2$.
$C_1^{(4)} = 700 + 300 + 600 + 900 + 600c = 2500 + 600c$
$C_1^{(3)} = 700 + 300 + 600 + 300c + C_4 = 2300 + 300c$
$C_1^{(2)} = 700 + 300 + +100c + C_3 = 2000 + 200c$
$C_1^{(1)} = 700 + C_2 = 2300 + 300c$
$C_1 = \min\{C_1^{(1)}, C_1^{(2)}, C_1^{(3)}, C_1^{(4)}\} = 2000$
as long as $c \geq -1.5$ and $c \leq 1/2$.
If $c < -1.5$ then, the optimal plan is to order 400 kilo before the first week. If $c > 1/2$, then the optimal plan would change if we ended up with zero inventory at the beginning of the second week, but this is not the case here. If $c > 4$, then the optimal plan is to order 100 kilo every week. (for $c = 4$ the cost of storing 100 kilo one week is the same as that of ordering)

3. (a) $P_i(n_i)$ = *the probability that there is no problem of type i if Frasse has applied n_i extra measures of type i*
Then $f(n_1, \cdots, n_N) = P(n_1)P(n_2) \cdot \ldots \cdot P(n_N)$ described the probability that there are no problems of any category.
The optimization problem is then

$$\begin{array}{rl} \max\limits_{n_1,\cdots,n_N} & f(n_1, \cdots, n_N) \\ \text{s.t.} & \sum_{i=1}^N C_i(n_i) = \sum_{I=1}^N c_i n_i \leq S \\ & n_i \in \{0, 1, 2, 3, 4, 5\} \text{ for } i = 1, \cdots, N. \end{array}$$

(b) Introduce the stage ℓ as the reduced problem when Frasse has only measures of type $\ell, \cdots, N$ to choose from. Introduce the state, s_ℓ = how many dollars Frasse has at stage ℓ, which we assume is non-negative. Let x_ℓ be the number of measures of type ℓ that Frasse decides to take. Then $s_{\ell+1} = s_\ell - c_\ell x_\ell$.
Let $f_\ell^*(s_\ell)$ be the optimal value of the reduced problem when the budget is s_ℓ.
The DynP recursion can be written as

$$f_\ell^*(s_\ell) = \max_{x_\ell = 0,\cdots,[s_\ell/c_\ell]} \left\{P_\ell(x_\ell) f_{\ell+1}^*(s_\ell - x_\ell c_\ell)\right\},$$

where $[\cdot]$ denotes the integer part of its argument, and the boundary condition is that $f_{N+1}^* = 1$, or $f_N^*(s_N) = P_N([s_N/c_N])$.
Assume $N = 2$, $k_1 = 1/10$, $k_2 = 1/20$, $p_1 = 0.1$, $p_2 = 0.2$, $c_1 = 3$, $c_2 = 2$ and $S = 5$.
Let $f_3^* = 1$.
Then $f_2^*(s_2) = P_2([s_2/c_2]) = P_2([s_2/2]) = p_2 e^{k_2[s_2/2]}$
Then $f_1^*(s_1) = \max_{x_1=0,\cdots,[s_1/c_1]} \left\{P_1(x_1) f_2^*(s_1 - x_1 c_1)\right\}$, and in particular for $s_1 = 5$

$$f_1^*(5) = \max_{x_1=0,1} \left\{P_1(x_1) f_2^*(5 - 3x_1)\right\} = p_1 p_2 \max\left\{\exp k_1 \cdot 0 + k_2 \cdot 2, \exp k_1 \cdot 1 + k_2 \cdot 1\right\},$$

so $f_1^*(5) = p_1 p_2 e^{3/20}$ since $k_1 + k_2 = 3/20 > 2k_2 = 2/20$.

(c) Rewrite as the minimization problem

$$\begin{aligned} \min_{n_1,\cdots,n_N} \quad & -\log f(n_1, \cdots, n_N) \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^N C_i(n_i) = \sum_{I=1}^N c_i n_i \leq S \\ & n_i \in \{0,1,2,3,4,5\} \text{ for } i = 1, \cdots, N. \end{aligned}$$

Let $g(n) = \sum_{I=1}^N c_i n_i$ which is a seperable increasing integer-convex function. Then

$$F(n) = -\log f(n) = -\log \prod_{i=1}^N P_i(n_i) = -\sum_{i=1}^N \log P_i(n_i) = -\sum_{i=1}^N \log p_i + k_i n_i$$

is a seperable decreasing integer-convex function.

Note that $\Delta F_i(x) = -k_i$ and $\Delta g_i(x) = c_i$, so the quotients $-\Delta F_i(x)/\Delta g_i(x) = k_i/c_i$ does not depend on x. Here $k_1/c_1 = 0.1/3 > k_2/c_2 = 0.05/2$ so the marginal effect is always larger for the measure of type 1. The efficient allocations are therefore, $(n_1 = 0, n_2 = 0)$, $(n_1 = 1, n_2 = 0)$, $(n_1 = 2, n_2 = 0)$, $(n_1 = 3, n_2 = 0)$ corresponding to the total costs 0, 3, 6, 9 USD.

4. (a) We need to keep track of the quality of the crop, define the state

$$s_k = \begin{cases} 1 & \text{if the crop is good year } k \\ 0 & \text{if the crop is bad year } k \end{cases}$$

Define the decisions

$$x_k = \begin{cases} 1 & \text{if Frasse uses pesticides year } k \\ 2 & \text{if Frasse uses manure year} k \end{cases}$$

The transition probabilities are $p_{ij}(k) =$ *the probability of jumping from state i to j if we make decision k.* Here $p_{11}(1) = 0.8$, $p_{10}(1) = 0.2$, $p_{01}(1) = 0.6$, $p_{00}(1) = 0.4$, $p_{11}(2) = 0.6$, $p_{10}(2) = 0.4$, $p_{01}(2) = 0.4$, $p_{00}(2) = 0.6$.

In the costs we include the revenue with negative sign.

Then the costs of making decision $x_k = 1$ is $C_{11} = 600 - 0.8 * 1200 - 0.2 * 600 = -480$ if $s_k = 1$ and $C_{01} = 600 - 0.6 * 1200 - 0.4 * 600 = -360$ if $s_k = 0$,

the cost of making decision $x_k = 2$ is $C_{12} = 100 - 0.6 * 2000 - 0.4 * 800 = -1420$ if $s_k = 1$ and $C_{02} = 100 - 0.4 * 2000 - 0.6 * 800 = -1180$ if $s_k = 0$.

Starting policy:

Always make decision $x_k = 1$.

Use the policy iteration algorithm. Let $v_0 = 0$, then the value determination equations

$$g + v_1 = -480 + 0.8 v_1 + 0.2 v_0$$

$$g + v_0 = -360 + 0.6 v_1 + 0.4 v_0$$

gives $g = -450$, $v_1 = -150$.

To find out if it is optimal we do one step of the policy iteration.
For $i = 1$ (The crop is good)

$$\min_{k=1,2}\{C_{1k} + (p_{11}(k)v_1 + p_{10}(k)v_0)\} =$$

$$= \min\{C_{11} + (p_{11}(1)v_1 + p_{10}(1)v_0), C_{12} + (p_{11}(2)v_1 + p_{10}(2)v_0),\}$$
$$= \min\{\underbrace{-480 + (0.8v_1 + 0.2v_0)}_{-600}, \underbrace{-1420 + (0.6v_1 + 0.4v_0)}_{-1510},\} = -1510(\neq g{+}v_1) \text{ for } k = 2.$$

For $i = 0$ (The crop is bad)

$$\min_{k=1,2}\{C_{0k} + (p_{00}(k)v_0 + p_{01}(k)v_1)\} =$$

$$= \min\{C_{01} + (p_{01}(1)v_1 + p_{00}(1)v_0), C_{02} + (p_{01}(2)v_1 + p_{00}(2)v_0),\}$$
$$= \min\{\underbrace{-360 + (0.6v_1 + 0.4v_0)}_{-480}, \underbrace{-1180 + (0.4v_1 + 0.6v_0)}_{-12400},\} = -1240(\neq g{+}v_2) \text{ for } k = 2.$$

So the policy to always use manure is better.
Solving the value determination equation again for the new policy Let $v_0 = 0$, then the value determination equations

$$g + v_1 = -1420 + 0.6v_1 + 0.4v_0$$

$$g + v_0 = -1180 + 0.4v_1 + 0.6v_0$$

gives $g = -1300$. The average gain is 850 USD per year.

(b) We need to keep track of the quality of the crop, and the number of consecutive years of ecological farming. Define the state $S_k = (s_k, \sigma_k)$ where s_k is as before and $\sigma_k =$ *the number of consecutive years of ecological farming.* Note that σ_k can take any (non-negative) integer value.

Define the decisions as before. The transition probabilities are $p_{imjn}(k) =$ *the probability of jumping from state* (i, m) *to* (j, n) *if we make decision* k. Here $p_{1m1n}(1) = 0.8$, $p_{1m0n}(1) = 0.2$, $p_{0m1n}(1) = 0.6$, $p_{0m0n}(1) = 0.4$, if $n = 0$, and otherwise it is zero, and $p_{1m1n}(2) = 0.6$, $p_{1m0n}(2) = 0.4$, $p_{0m1n}(2) = 0.4$, $p_{0m0n}(2) = 0.6$. if $n = m + 1$, and otherwise it is zero.

Let $C_{ijk} =$ *the cost when in state* $s = i$, $\sigma = k$ *making decision* $x = j$. Then $C_{ijk} = C_{ij} - 100k$ if $j = 2$ and $C_{ijk} = C_{ij}$ if $j = 1$.

For the Markov decision Process algorithm to work, the system has to tend to a stationary condition, but for this problem the extra state is not finite and no state will actually be recurrent for the optimal strategy.