KTH Mathematics

Suggested solutions for the exam in SF2863 Systems Engineering.
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1. Without Heathcliff, for the receptionist $\rho_R = \lambda_R/\mu_R = 1/10$ we get

$$W_R = \frac{L_R}{\lambda_R} = \frac{1}{\lambda_R}\frac{\rho_R}{1-\rho_R} = 1/9$$

and for Frasse $\rho_F = \lambda_F/\mu_F = 1/1.2$ we get

$$W_F = \frac{L}{\lambda} = \frac{1}{\lambda_F}\frac{\rho_F}{1-\rho_F} = 5$$

The average passing time through the system is then $W_1 = 5 + 1/9$.

With Heathcliff, for the receptionist we get the same. For Heathcliff $\rho_H = (1-p)/2$, and

$$W_H = \frac{L_H}{\lambda_H} = \frac{1}{\lambda_H}\frac{\rho_H}{1-\rho_H} = \frac{1}{\lambda_H}\frac{(1-p)/2}{1-(1-p)/2} = \frac{1}{1+p}$$

and for Frasse we have now $\lambda_F = (p + 1/2(1-p))\cdot 1 = (1+p)/2$. $\rho_F = \lambda_F/\mu_F = (1+p)/2.4$ we get

$$W_F = \frac{L_F}{\lambda_F} = \frac{1}{\lambda_F}\frac{\rho_F}{1-\rho_F} = \frac{2}{1.4-p}.$$

The average passing time through the system is then $W_2(p) = 1/9 + (1-p)W_H + [(1-p)/2 + p]W_F = 1/9 + (1-p)/(1+p) + (1+p)/(1.4-p)$. The minimum is achieved for $\hat{p} = 2.4\sqrt{30} - 13$ where the derivative is equal to 0.

The probability that the $M|M|1$ system is empty is $P_0 = 1 - \rho$.

Without Heathcliff, using independence, we get $(1-1/10)(1-1/1.2) = 0.15$.

With Heathcliff, using independence, we get $(1-1/10)(1-(1-p)/2)(1-(1+p)/2.4 = 9/40(1-p^2)$.

The savings per hour for the company is $(W_1 - W_2(\hat{p}))\lambda \cdot 100 = 5 - (1-\hat{p})/(1+\hat{p}) - (1+\hat{p})/(1.4-\hat{p}) = 334$ dollars . So employing Heathcliff would save them money. (Unless the company is paying a salary that is above the value their personel is generating)

2. Define the variables.

Let s_n = number of laps skied with current skis as lap number n is initiated.

Let the decision $x_n = 1$ if the skies are changed before starting lap n, and the decision $x_n = 0$ if the skies are *not* changed before starting lap n.

Let $V_n(s_n)$ =optimal time to goal if at start of lap n with s_n laps skied with current skies.

Let $V_n(s_n, x)$ =optimal time to goal if at start of lap n with s_n laps skied with current skies, and decision x is taken about the ski-change before lap n.

Then

$$V_n(s) = \min_{x \in \{0,1\}} V_n(s,x) = \min\{25 + 1/2 + V_{n+1}(1), 25 + s/3 + V_{n+1}(s+1)\}$$

If the skier does no change skis the lap time increases with 1/3 minute for each lap skied with the current skies. If the skier changes skis the lap time increases with 1/2 minute for the actual change.

Where we also used that $s_{n+1} = s_n + 1$ if the skier does not change skis, and $s_{n+1} = 1$ if the skier changes skies before going out for lap n.

We have that $V_6(s) = 0$. Then

$$V_5(s) = \min\{25 + 1/2, 25 + s/3\} = \begin{cases} 25 + \frac{1}{2} & \text{if } s \geq 2 \\ 25 + \frac{s}{3} & \text{if } s \leq 1 \end{cases}$$

We use here that $s \geq 1$ for $n \geq 2$.

$$V_4(s) = \min\{25 + 1/2 + V_5(1), 25 + s/3 + V_5(s+1)\}$$

$$= \min\{25 + 1/2 + 25 + 1/3, 25 + s/3 + 25 + 1/2\} = 50 + \frac{5}{6}$$

We use here that $s \geq 1$ for $n \geq 2$.

$$V_3(s) = \min\{25 + 1/2 + V_4(1), 25 + s/3 + V_4(s+1)\}$$

$$= \min\{25 + 1/2 + 50 + 5/6, 25 + s/3 + 50 + 5/6\} = \begin{cases} 75 + \frac{8}{6} & \text{if } s \geq 2 \\ 75 + \frac{7}{6} & \text{if } s = 1 \end{cases}$$

We use here that $s \geq 1$ for $n \geq 2$.

$$V_2(s) = \min\{25 + 1/2 + V_3(1), 25 + s/3 + V_3(s+1)\}$$

$$= \min\{25 + 1/2 + 75 + 7/6, 25 + s/3 + 75 + 8/6\} = 100 + \frac{10}{6}$$

Finally,

$$V_1(0) = \min\{25 + 1/2 + V_2(1), 25 + s/3 + V_2(s+1)\}_{s=0}$$

$$= \min\{25 + 1/2 + 100 + 10/6, 25 + 0/3 + 100 + 10/6\} = 125 + \frac{10}{6}.$$

What is the optimal strategy.

With fresh skis from the start $s_1 = 0$, no change is done for the first lap.
For the second lap, $s_2 = 1$, change.
For the third lap, $s_3 = 1$, no change.
For the fourth lap, $s_4 = 2$, change.
For the fifth lap, $s_5 = 1$, no change.

There is an alternative solution which is equally good, if for the second lap there is no change and then change, no change and change.

3. Let d_i denote the number of training resources that are assigned to athlete i, $i = 1, 2, 3, 4.$.

Define functions f and g that we want to minimize, with the right properties. Maximizing performance is the same as minimizing the total finish time. Let $f(d_1, d_2, d_3, d_4) = \left[(d_1^2 - 10d_1 + 30) + (d_2^2 - 11d_2 + 36) + (d_3^2 - 12d_3 + 44) + (d_4^2 - 13d_4 + 50)\right]$ and $g(d_1, d_2, d_3, d_4) = \sum_{i=1}^{4} d_i$. Clearly g is a separable function, increasing and integer-convex.

The continuous version of function f has a gradient $\nabla f = (2d_1 - 10, 2d_2 - 11, 2d_3 - 12, 2d_4 - 13)$ which has negative elements for d_i less than 8, which makes the function decreasing. Furthermore, f is seperable, $f = f_1(d_1) + f_2(d_2) + f_3(d_3) + f_4(d_4)$, and the functions f_i are quadratic functions which are convex, since the Hessian is 2 times the identity matrix.

(Note that $\Delta f_i(x) \neq f_i'(x)$)

$-\Delta f$	$-\Delta f_1$	$-\Delta f_2$	$-\Delta f_3$	$-\Delta f_4$
$d_i = 1$	9	10	11	12
$d_i = 2$	7	8	9	10
$d_i = 3$	5	6	7	8
$d_i = 4$	3	4	5	6

Since the values are decreasing in each column the function f is integer-convex for the tabulated values.

Note that $\Delta g_i = 1$ for all i.

We can then apply the Marginal Allocation algorithm.

Note that the quotients $-\Delta f_i(d)/\Delta g_i(d) = \Delta f_i(d_i)$ are given in the table above.

The efficient allocations are therefore,
$S^{(0)} = (s_1 = 0, s_2 = 0, s_3 = 0, s_4 = 0),\ f(s^{(0)}) = 160,\ g(s^{(0)}) = 0$
$S^{(1)} = (s_1 = 0, s_2 = 0, s_3 = 0, s_4 = 1),\ f(s^{(1)}) = 148,\ g(s^{(1)}) = 1$
$S^{(2)} = (s_1 = 0, s_2 = 0, s_3 = 1, s_4 = 1),\ f(s^{(2)}) = 137,\ g(s^{(2)}) = 2$
$S^{(3)} = (s_1 = 0, s_2 = 1, s_3 = 1, s_4 = 1), f(s^{(3)}) = 127,\ g(s^{(3)}) = 3$
$S^{(4)} = (s_1 = 0, s_2 = 1, s_3 = 1, s_4 = 2),\ f(s^{(4)}) = 117,\ g(s^{(4)}) = 4$
$S^{(5)} = (s_1 = 1, s_2 = 1, s_3 = 1, s_4 = 2), f(s^{(5)}) = 108,\ g(s^{(5)}) = 5$
$S^{(6)} = (s_1 = 1, s_2 = 1, s_3 = 2, s_4 = 2),\ f(s^{(6)}) = 99,\ g(s^{(6)}) = 6$

For $s^{(3)}$ there is a an alternative solution with $s_2 = 0$ and $s_4 = 2$.
For $s^{(5)}$ there is a an alternative solution with $s_1 = 0$ and $s_3 = 2$.

4. (a) Let D be the stochastic variabel denoting the demand, and let B denote the number of tickets that Frasse orders.
The profit of Frasse will be

$$100 \min(D, B) + 10 \max(B - D, 0) - 60B$$

and the expected profit is then

$$100\mathbf{E}[D - (D - B)^+] + 10\mathbf{E}[(B - D)^+] - 60B.$$

Expected cost (disregarding $100\mathbf{E}[D]$ which does not depend on B) is

$$C(B) = 100\mathbf{E}(D-B)^+] - 10\mathbf{E}[(B-D)^+] + 60B.$$

Identifying $c = 60$, $p = 100$, $h = -10$, then

$$\frac{p-c}{p+h} = \frac{100-60}{100-10} = \frac{4}{9}$$

The cumulative distribution function for D is $F_D(t) = P(D \leq t) = (t-39)/61$
We should find the smallest B^* such that $F_D(B^*) \geq 4/9$, which gives $B^* = 67$.

(b) This a deterministic periodic review model, with a small modification. If we order more than 4 cars, the ordering cost goes up
We still have

$$C_i = \min_j \{C_i^{(j)} | i \leq j \leq N\}$$

where

$$C_i^{(j)} = C_{j+1} + K \cdot N + h(r_{i+1} + 2r_{i+2} + \cdots + (j-i)r_j),$$

$K = 5$, $h = 1$ and $N = \mathit{ceiling}\,\{(r_i + r_{i+1} + r_{i+2} + \cdots + r_j)/4\}$.
When
$C_4^{(4)} = 5$
then
$C_4 = \min_{j=4} C_4^{(j)} = \min\{C_4^{(4)}\} = 5$
When
$C_3^{(3)} = 5 + 5 = 10$
$C_3^{(4)} = 5 + 2 = 7$
then
$C_3 = \min_{j=3,4} C_3^{(j)} = \min\{C_3^{(3)}, C_3^{(4)}\} = 7$
When
$C_2^{(2)} = 5 + 7 = 12$
$C_2^{(3)} = 5 + 1(1) + 5 = 11$
$C_2^{(4)} = 10 + 1(1 + 2 \cdot 2) = 15$
then
$C_2 = \min_{j=2,\cdots,4} C_2^{(j)} = \min\{C_2^{(2)}, C_2^{(3)}, C_2^{(4)}\} = 11$
When
$C_1^{(1)} = 5 + 11 = 16$
$C_1^{(2)} = 10 + 1(3) + 7 = 20$
$C_1^{(3)} = 10 + 1(3 + 2 \cdot 1) + 5 = 20$
$C_1^{(4)} = 15 + 1(3 + 2 \cdot 1 + 3 \cdot 2) = 26$
then
$C_1 = \min_{j=1,\cdots,4} C_1^{(j)} = \min\{C_1^{(1)}, C_1^{(2)}, C_1^{(3)}, C_1^{(4)}\} = 16$
So the optimal strategy is: First week order 3, second week order 4 third week order 0, fourth week order 2. Total cost is then 16.000 dollars.

A counter-example showing that it is not always optimal to order only when inventory level is zero.

Let $d_1 = 5$ and $d_2 = 5$.

Then ordering 10 the first day costs $15 + 1 \cdot 5 = 20$. Ordering 5 the first day and 5 the second day costs $10 + 10 = 20$. These are the only two alternatives for ordering when the inventory is at zero.

Ordering 6 the first day and 4 the second day costs $10 + 1 \cdot 1 + 5 = 16$ and is less expensive than the above alternative.

5. We need to keep track of if the sale went better or worse than expected last year. Let the state be

$$s_k = \begin{cases} 1 & \text{if the sale went better than expected the year before year } k \\ 0 & \text{if the sale went worse than expected the year before year } k \end{cases}$$

Define the decisions

$$x_k = \begin{cases} 1 & \text{if Frasse decides to use a high starting price year } k \\ 0 & \text{if Frasse decides to use a low starting price year } k \end{cases}$$

The transition probabilities are
$p_{ij}(x) =$ *the probability of jumping from state i to j if we make decision x.*

$$P(x = 1) = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}$$

$$P(x = 0) = \begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix}$$

Let the costs be the happiness and maximize instead of minimize.

Let $q_{ij}(x) =$ expected cost incurred when the state is in state i decision x is made and the system evolves to state j.

$$Q(x = 0) = \begin{bmatrix} 1/2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$Q(x = 1) = \begin{bmatrix} 1 & 2 \\ 1/2 & 3 \end{bmatrix}$$

Then the expected "cost" of making decision x_k at state s_k is $C_{s_k,x_k} = \sum_{j=0}^{3} q_{s_k,j} p_{s_k,j}(x_k)$.

$$C_{00} = q_{00}(0)p_{00}(0) + q_{01}(0)p_{01}(0) = 4/5$$

$$C_{01} = q_{00}(1)p_{00}(1) + q_{01}(1)p_{01}(1) = 7/5$$

$$C_{10} = q_{10}(0)p_{10}(1) + q_{11}(0)p_{11}(0) = 9/5$$

$$C_{11} = q_{10}(1)p_{10}(1) + q_{11}(1)p_{11}(1) = 7/4$$

Starting policy:
If $s_k = 0$, make decision $x_k = 1$.
If $s_k = 1$, make decision $x_k = 1$.

Use the policy iteration algorithm. Let $v_1 = 0$, then the value determination equations

$$g + v_0 = 7/5 + 0.4v_0 + 0.6v_1$$

$$g + v_1 = 7/4 + 0.2v_0 + 0.8v_1$$

gives $g = 14/9$, $v_0 = -7/18$ and $v_1 = 0$.

To find out if it is optimal we do one step of the policy iteration.

For $i = 0$

$$\max_{k=1,2}\{C_{0k} + (p_{00}(k)v_0 + p_{01}(k)v_1)\} =$$

$$= \max\{C_{00} + (p_{00}(0)v_0 + p_{01}(0)v_1), C_{01} + (p_{00}(1)v_0 + p_{01}(1)v_1)\}$$

$$= \max\{\underbrace{4/5 + 0.4 * (-7/18)}_{0.64}, \underbrace{8/5 + 0.6 * (-7/18)}_{1.37}\} = 1.37 \text{ for } k = 1.$$

For $i = 1$

$$\min_{k=0,1}\{C_{1k} + (p_{10}(k)v_0 + p_{11}(k)v_1)\} =$$

$$= \max\{C_{10} + (p_{10}(0)v_0 + p_{11}(0)v_1), C_{11} + (p_{10}(1)v_0 + p_{11}(1)v_1)\}$$

$$= \max\{\underbrace{9/5 + 0.2 * (-7/18)}_{1.72}, \underbrace{7/4 + 0.5 * (-7/18)}_{1.56}, \} = 1.72 \text{ for } k = 0.$$

The starting policy is not optimal, it is better to use a low starting price if the past year gave a higher than expected income.