KTH Mathematics

Suggested solutions for the exam in SF2863 Systems Engineering.
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1. The intensity of arrivals to the arena is $\lambda_A = 1000$ per hour, the service station is a $M|M|1$ system with service intensity $\mu_A = 60 \cdot 60/3 = 1200$ per hour. The expected time for them to pass through the entrance check is $W_A = L_A/\lambda_A = \rho_A/(1-\rho_A)/1000 = 1/200$ hours, where $\rho_A = \lambda_A/\mu_A = 1/1.2 = 5/6$ (+ 3 seconds for Frasse to pass through).

The souvenir shop is a $M|M|1$ system with arrival intensity λ_1 and service intensity $\mu_1 = 180$ customers per hour. The snacks shop is a $M|M|2$ system with arrival intensity λ_2 and service intensity $\mu_2 = 120$ customers per hour and per server.

Assuming that the shops form a Jackson network we obtain the conservation of flow equations

$$\lambda_1 = 0.1\lambda_A + 0.3\lambda_2, \quad \lambda_2 = 0.2\lambda_A + 0.2\lambda_1.$$

And then $\lambda_1 = 160/0.94$ and $\lambda_2 = 220/0.94$.

The expected time to go through the souvenir shop once is

$$W_1 = L_1/\lambda_1 = \rho_1/(1-\rho_1)0.94/160 = 47/460, \quad \rho_1 = \lambda_1/\mu_1 = 160/0.94/180 = 400/423$$

The expected time to go through the snack shop once is

$$W_2 = L_2/\lambda_2 = 2*\rho_2/(1-\rho_2^2)0.94/220 = 867/5101, \quad \rho_1 = \lambda_1/\mu_1 = 160/0.94/180 = 400/423$$

From

$$V_1 = W_1 + 0.2V_2, \quad V_2 = W_2 + 0.3V_1$$

we get $V_1 = 0.145$ and $V_2 = 0.213$. Expected time to pass through the shops is $0.1V_1 + 0.2V_2 = 0.0572$, i.e. 3.4 minutes.

The total expected time is then

$$W_A + \underbrace{1/12}_{5\text{min}} + 0.0572 + \underbrace{1/30}_{2\text{min}} = 0.18 \quad \text{hours.}$$

2. Define the variables.

Let s_n = number of more points than the other team after game n.
Starting at $s_0 = 0$.

Let the decision $x_n = 0$ if Zlatan is not provoced in game n,
and the decision $x_n = 1$ if Zlatan is provoced in game n,

Let $V_n(s_n)$ = probability of advancing if after game n PSG has s_n points more than the other team and an optimal provocation strategy is applied.

Here,

$$V_2(s_2) = \begin{cases} 1 & \text{if } s_2 > 0, \\ 1/2 & \text{if } s_2 = 0, \\ 0 & \text{if } s_2 < 0 \end{cases}$$

Let $P_{ij}(x)$ be the probability of going from state i to j using decision x.

Then

$$V_n(s) = \max_{x \in \{0,1} \mathrm{E}V_n(s,x) = \max \left\{ p_{s,s-3}(x)V_{n+1}(s-3) + p_{s,s}(x)V_{n+1}(s) + p_{s,s+3}(x)V_{n+1}(s+3) \right\},$$

Then

$$V_1(s) = \max_{x \in \{0,1\}} V_1(s,x) = \max_{x \in \{0,1\}} \left\{ p_{s,s-3}(x)V_2(s-3) + p_{s,s}(x)V_2(s) + p_{s,s+3}(x)V_2(s+3) \right\},$$

so

$$V_1(3) = \max_{x \in \{0,1\}} V_1(3,x)$$

$$= \max_{x \in \{0,1\}} \left\{ p_{3,0}(0)V_2(0) + p_{3,3}(0)V_2(3) + p_{3,6}(0)V_2(6), p_{3,0}(1)V_2(0) + p_{3,3}(1)V_2(3) + p_{3,6}(1)V_2(6) \right\},$$

$$= \max \left\{ 0.2 \cdot 1/2 + 0.3 \cdot 1 + 0.5 \cdot 1, 0.3 \cdot 1/2 + 0.1 \cdot 1 + 0.6 \cdot 1 \right\} = 0.9,$$

for decision $x = 0$.

$$V_1(0) = \max_{x \in \{0,1\}} V_1(0,x)$$

$$= \max_{x \in \{0,1\}} \left\{ p_{0,-3}(0)V_2(-3) + p_{0,0}(0)V_2(0) + p_{0,3}(0)V_2(3), p_{0,-3}(1)V_2(-3) + p_{0,0}(1)V_2(0) + p_{0,3}(1)V_2(3) \right\},$$

$$= \max \left\{ 0.3 \cdot 0 + 0.3 \cdot 1/2 + 0.4 \cdot 1, 0.4 \cdot 0 + 0.1 \cdot 1/2 + 0.5 \cdot 1 \right\} = 0.55,$$

for decision $x = 0$.

$$V_1(-3) = \max_{x \in \{0,1\}} V_1(-3,x)$$

$$= \max_{x \in \{0,1\}} \left\{ p_{-3,-6}(0)V_2(-6) + p_{-3,-3}(0)V_2(-3) + p_{-3,0}(0)V_2(0), p_{-3,-6}(1)V_2(-6) + p_{-3,-3}(1)V_2(-3) + p_{-3,0}(1)V_2(0) \right\},$$

$$= \max \left\{ 0.3 \cdot 0 + 0.4 \cdot 0 + 0.3 \cdot 1/2, 0.4 \cdot 0 + 0.2 \cdot 0 + 0.4 \cdot 1/2 \right\} = 0.2,$$

for decision $x = 1$.

The two teams starts with zero points.

$$V_0(0) = \max_{x \in \{0,1\}} V_0(0,x)$$

$$= \max_{x \in \{0,1\}} \left\{ p_{0,-3}(0)V_1(-3) + p_{0,0}(0)V_1(0) + p_{0,3}(0)V_1(3), p_{0,-3}(1)V_1(-3) + p_{0,0}(1)V_1(0) + p_{0,3}(1)V_1(3) \right\},$$

$$= \max \left\{ 0.4 \cdot 0 + 0.3 \cdot 1/2 + 0.3 \cdot 1, 0.5 \cdot 0 + 0.1 \cdot 1/2 + 0.4 \cdot 1 \right\} = 0.515,$$

for decision $x = 0, 1$.

So the optimal value is 0.515 probability for Chelsea to advance.

First game it does not matter if they provoce or not.

If they win the first game, they should not provoke in the second.

If they draw the first game, it does not matter if they provoke or not in the second.

If they lose the first game, they should provoke in the second.

3. Let s_i denote the number of stars the player at position i should have, $i = 1, 2, 3.$.

Define functions f and g that we want to minimize, with the right properties. Maximizing the utility is the same as minimizing minus the utility. Let $U_i(s_i)$ be defined by the table. Let $f(s_1, s_2, s_3) = -[u_1(s_1) + u_2(s_2) + u_3(s_3)]$ Clearly f is a separable function, and decreasing.

Integer-convexity follows by considering $\Delta u_i(1)$, i.e.

Δf	$-\Delta u_1$	$-\Delta u_2$	$-\Delta u_3$
$s_i = 1$	-3	-2	-5
$s_i = 2$	-2	-1	-3

which are increasing.

We want to minimize the cost, let $g(s_1, s_2, s_3) = \sum_{i=1}^{3} c_i(s_i)$. Clearly g is a separable function, increasing, Integer-convexity follows by considering $\Delta c_i(1)$, i.e.

Δg	Δc_1	Δc_2	Δc_3
$s_i = 1$	0.3	0.3	0.3
$s_i = 2$	0.4	0.3	0.4

which are increasing.

We can then apply the Marginal Allocation algorithm.

$-\Delta f/\Delta g$	$\frac{-\Delta f_1}{\Delta g_1}$	$\frac{-\Delta f_2}{\Delta g_2}$	$\frac{-\Delta f_3}{\Delta g_3}$
$s_i = 1$	10	6 2/3	16 2/3
$s_i = 2$	5	3 1/3	7 1/2

The efficient allocations are therefore,
$S^{(0)} = (s_1 = 1, s_2 = 1, s_3 = 1)$, $f(s^{(0)}) = -15$, $g(s^{(0)}) = 1.1$
$S^{(1)} = (s_1 = 1, s_2 = 1, s_3 = 2)$, $f(s^{(1)}) = -20$, $g(s^{(1)}) = 1.4$
$S^{(2)} = (s_1 = 2, s_2 = 1, s_3 = 2)$, $f(s^{(2)}) = -23$, $g(s^{(2)}) = 1.7$
$S^{(3)} = (s_1 = 2, s_2 = 1, s_3 = 3)$,$f(s^{(3)}) = -26$, $g(s^{(3)}) = 2.1$

4. We need to keep track of if Zlatan is starts the year with a new contract, on the second year of a contract or on his third year.

$$s_k = \begin{cases} 1 & \text{is on the first year of a contract at start of year } k \\ 2 & \text{is on the second year of a contract at start of year } k \\ 3 & \text{is on the third year of a contract at start of year } k \end{cases}$$

Define the decisions

$$x_k = \begin{cases} 1 & \text{if Zlatan decides to stay with the contract after year } k \\ 0 & \text{if Zlatan decides to get a new contract after year } k \end{cases}$$

The state update equation is $s_{k+1} = s_k + 1$ if $x_k = 1$, and is $s_{k+1} = 1$ if $x_k = 0$.

The transition probabilities are
$p_{ij}(x) =$ *the probability of jumping from state i to j if we make decision x.*

If he stays:

$$P(x=1) = \begin{bmatrix} p_{11}(1) & p_{12}(1) & p_{13}(1) \\ p_{21}(1) & p_{22}(1) & p_{23}(1) \\ p_{31}(1) & p_{32}(1) & p_{33}(1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

If he decides to change:

$$P(x=0) = \begin{bmatrix} p_{11}(0) & p_{12}(0) & p_{13}(0) \\ p_{21}(0) & p_{22}(0) & p_{23}(0) \\ p_{31}(0) & p_{32}(0) & p_{33}(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The different policies are

Policy 1
In state 1: stay, In state 2: stay, In state 3: change.
Policy 2
In state 1: stay, In state 2: change, In state 3: change.
Policy 3
In state 1: change, In state 2: stay, In state 3: change.
Policy 4
In state 1: change, In state 2: change, In state 3: change.

For policy 1, all states are periodic with period 3.

For policy 2, states 1 and 2 are periodic with period 2, and state 3 is transient.

For policy 3 and 4, state 1 is absorbing, and state 2 and 3 are transient.

Then the expected "cost" of making decision x_k at state s_k is C_{s_k,x_k}, here

$$C_{10} = 60, \quad C_{11} = 40,$$
$$C_{20} = 50, \quad C_{21} = 30,$$
$$C_{30} = 100,$$

Starting policy:
If $s_k = 1$, make decision $x_k = 1$.

If $s_k = 2$, make decision $x_k = 1$.
If $s_k = 3$, he can only make decision $x_k = 1$.

Use the policy iteration algorithm. The value determination equations

$$V_1 = C_{11} + \alpha(p_{11}(1)V_1 + p_{12}(1)V_2 + p_{13}(1)V_3)$$
$$V_2 = C_{21} + \alpha(p_{21}(1)V_1 + p_{22}(1)V_2 + p_{23}(1)V_3)$$
$$V_3 = C_{30} + \alpha(p_{31}(0)V_1 + p_{32}(0)V_2 + p_{33}(0)V_3)$$

are

$$V_1 = 40 + 0.5(0V_1 + 1V_2 + 0V_3)$$
$$V_2 = 30 + 0.5(0V_1 + 0V_2 + 1V_3)$$
$$V_3 = 100 + 0.5(1V_1 + 0V_2 + 0V_3)$$

gives $V_1 = 640/7$, $V_2 = 720/7$ and $V_3 = 1020/7$.

To find out if it is optimal we do one step of the policy iteration.

For $i = 1$

$$\max_{k=0,1}\{C_{1k} + \alpha(p_{11}(k)V_1 + p_{12}(k)V_2 + p_{13}(k)V_3)\} =$$

$$= \max\{C_{10} + 0.5(p_{11}(0)V_1 + p_{12}(0)V_2 + p_{13}(0)V_3), C_{11} + 0.5(p_{11}(1)V_1 + p_{12}(1)V_2 + p_{13}(1)V_3)\}$$
$$= \max\{\underbrace{60 + 0.5(640/7)}_{740/7}, \underbrace{40 + 0.5(720/7)}_{640/7}\} = 740/7 \text{ for } k = 0.$$

For $i = 2$

$$\max_{k=0,1}\{C_{2k} + \alpha(p_{21}(k)V_1 + p_{22}(k)V_2 + p_{23}(k)V_3)\} =$$

$$= \max\{C_{20} + 0.5(p_{21}(0)V_1 + p_{22}(0)V_2 + p_{23}(0)V_3), C_{21} + 0.5(p_{21}(1)V_1 + p_{22}(1)V_2 + p_{23}(1)V_3)\}$$
$$= \max\{\underbrace{50 + 0.5(640/7)}_{670/7}, \underbrace{30 + 0.5(1020/7)}_{720/7}\} = 720/7 \text{ for } k = 1.$$

The starting policy is not optimal, to maximize the discounted future income He should change the contract every year. If he starts on the second year of a contract he should stay until the third and then change every year.

5. (a) This can be modelled as an economic order quantity model, EOQ model.
The demand is $d = 20$ shirts per day. The holding cost is $h = 0.4$ Euro per package and day. The ordering cost is $K = 100$ Euro.
Then the quantity to be ordered is

$$\hat{Q} = \sqrt{\frac{2dK}{h}} = 100 \text{ shirts.}$$

He should order 100 shirts each time and he should order every 5 day.

(b) The change of price does not affect the location of the point where the derivative is zero. This point will still be the minimum if the value at the discontinuity introduced by the discount is not lower.

The total cost per time unit is determined in the book at page 807,

$$T(Q) = \frac{dK}{Q} + dc + \frac{hQ}{2}$$

So we have to compare $T(100)$ and $T(500)$

$$\frac{20 \cdot 100}{100} + 20 \cdot 5 + \frac{0.4 \cdot 100}{2} = 140 > 164 = \frac{20 \cdot 100}{500} + 20 \cdot 3 + \frac{0.4 \cdot 500}{2}$$

Ordering the larger quantity is more expensive per day than the old strategy, so do not change strategy.

(c) We consider the expected value of a loss of shirts as a cost, i.e.

$$\int_0^{Q/d} c(Q - d \cdot t)pdt = \frac{cpQ^2}{2d}$$

is the additional cost on average for sometimes losing the value of wrong club shirts.

As you can see this can be seen as modifying the holding cost.

The new optimal order quantity would be

$$\hat{Q}^* = \sqrt{\frac{2dK}{h + cp}} \approx 97 \text{ shirts.}$$

So the change would be small.