KTH Mathematics

Suggested solutions for the exam in SF2863 Systems Engineering. January 11, 2016

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1. Let n_i denote the level of action for feature i, where $i = 1, 2, 3.$.

Define functions f and g that we want to minimize, with the right properties. Frasse wants to minimze the environmental effects. Let $E_i(s_i)$ be defined by the table. Let $f(s_1, s_2, s_3) = [E_e(s_1) + E_c(s_2) + E_m(s_3)]$ Clearly f is a separable function, and decreasing.

Integer-convexity follows by considering $\Delta f_i(1)$, i.e.

Δf	ΔE_E	ΔE_c	ΔE_m
$s_i = 0$	-5	-6	-3
$s_i = 1$	-2	-3	-2
$s_i = 2$	-1	-2	-1

which are increasing.

We want to minimize the cost, let $g(s_1, s_2, s_3) = c_E(s_1) + c_c(s_2) + c_m(s_3)$. Clearly g is a separable function, increasing, Integer-convexity follows by considering $\Delta c_i(1)$, i.e.

Δg	Δc_E	Δc_c	Δc_m
$s_i = 0$	2	2	4
$s_i = 1$	8	8	6
$s_i = 2$	12	14	14

which are increasing.

We can then apply the Marginal Allocation algorithm.

$-\Delta f/\Delta g$	$\frac{-\Delta f_1}{\Delta g_1}$	$\frac{-\Delta f_2}{\Delta g_2}$	$\frac{-\Delta f_3}{\Delta g_3}$
$s_i = 0$	$5/2 = 2.5$	$6/2 = 3$	$3/4 = 0.75$
$s_i = 1$	$2/8 = 0.25$	$3/8 = 0.375$	$2/6 = 0.33$
$s_i = 2$	$1/12 = 0.083$	$2/12 = 0.14$	$1/14 = 0.07$

The efficient allocations are therefore,

$S^{(0)} = (s_1 = 0, s_2 = 0, s_3 = 0), \quad f(s^{(0)}) = 45, \quad g(s^{(0)}) = 44$
$S^{(1)} = (s_1 = 0, s_2 = 1, s_3 = 0), \quad f(s^{(1)}) = 39, \quad g(s^{(1)}) = 46$
$S^{(2)} = (s_1 = 1, s_2 = 1, s_3 = 0), \quad f(s^{(2)}) = 34, \quad g(s^{(2)}) = 48$
$S^{(3)} = (s_1 = 1, s_2 = 1, s_3 = 1), \quad f(s^{(3)}) = 31, \quad g(s^{(3)}) = 52$
$S^{(4)} = (s_1 = 1, s_2 = 2, s_3 = 1), \quad f(s^{(3)}) = 28, \quad g(s^{(3)}) = 60$

2. The intensity of arrivals of patent applications (from outside) is $\lambda_A = 10$ per month, the service station is a $M|M|1$ system with service intensity $\mu_A = 16$ per month.

Let λ_B be the arrival rate (with feedback) to the patent lawyer, and the patent office. From $\lambda_B = 0.3\lambda_B + \lambda_A$, we get $\lambda_B = 100/7$.

The expected number of patent papers at the patent lawyer is $L_A = \rho_A/(1-\rho_A) = 100/12 = 25/3$ papers, and the expected time for a paper to pass through the patent lawyer (once) is $W_A = L_A/\lambda_B = 7/12$ months, where $\rho_A = \lambda_B/\mu_A = 100/7/16 = 100/112 = 25/28$.

The expected number of patent papers at the patent office is $L_B = 2\rho_B/(1-\rho_B^2) = 35/12$ papers, where $\rho_B = \lambda_B/\mu_B/2 = 5/7$, and $\mu_B = 10$.

Expected total number of papers being handled is $L = L_A + L_B = 25/3 + 35/12 = 45/4$. From global Little's formula we have that the average time for handling a patent paper is $W = L/\lambda_A = 45/40$ months.

The probability that a patent paper is finally rejected is $0.2/(0.5+0.2) = 2/7$ and the probability that a patent paper is finally accepted is $0.5/(0.5+0.2) = 5/7$.

The average number N of times that a paper is handled by the patent lawyer, satisfies $N = 1 + 0.3N$, *i.e.*, $N = 1/0.7 = 10/7$, and the average time it is "in service" is $10/7(1/\mu_A + 1/\mu_B) = 13/56$, so the average total cost is $13/56 * 500 * 24 = 19500/7$ dollars.

3. Define the reduced problems

$$(\mathcal{P}_n) \quad \left[\begin{array}{ll} \max\limits_{x_n,\cdots,x_N} & \sum_{i=n}^N x_i p_i \\ \text{s.t} & \sum_{i=n}^N x_i \ell_i \le s_n \\ & x_i \in \{0,1,2,\cdots\} \text{ for } i = 1,\cdots,N. \end{array} \right]$$

where $s_n =$ the length of the log that is to be cut in pieces of lengths $\ell_n, \cdots, \ell_N$. The original problem is $(\mathcal{P}_1)$ starting at $s_1 = L$.

Let $f_n^*(s_n) =$ the optimal value of the optimization problem $(\mathcal{P}_n)$, and let $f_n^*(s_n, z) =$ the optimal value of the optimization problem $(\mathcal{P}_n)$ when z pieces of length ℓ_n are cut and the rest of the log is cut optimally.

Then

$$f_n^*(s_n) = \max_z \{ f_n^*(s_n, z) |\, s_n - z\ell_n \ge 0\} = \max_z \{ zp_n + f_n^*(s_n - z\ell_n) |\, s_n - z\ell_n \ge 0\}.$$

We now solve it for $L = 4$, that is we need to solve for $f_n^*(s_n)$ for $s_n \le 4$ and $n = 2, 3, 4$ to obtain $f_1^*(4)$.

Let $f_5^* \equiv 0$, then $f_4^*(s_4) = 9$ if $s_4 = 4$ and 0 otherwise.

For $n = 3$, $f_3^*(s_3) = \max_z \{ 7z + f_4^*(s_3 - 3z) |\, s_3 - 3z \ge 0\}$

s_3	0	1	2	3	4
f_3^*	0	0	0	7	9
$\hat{z}$	0	0	0	1	0

For $n = 2$, $f_2^*(s_2) = \max_z \{5z + f_3^*(s_2 - 2z) | \, s_2 - 2z \geq 0\}$

s_2	0	1	2	3	4
f_2^*	0	0	5	7	10
$\hat{z}$	0	0	0	0	2

For $n = 1$, $f_1^*(4) = \max_z \{2z + f_2^*(4 - z) | \, 4 - z \geq 0\}$, so we compare

$$0 + f_2^*(4) = 10, \quad 2 + f_2^*(3) = 9, \quad 4 + f_2^*(2) = 9, \quad 6 + f_2^*(1) = 6, \quad 8 + f_2^*(0) = 8,$$

which acchieves the maximal value 10 by choosing $x_1 = 0$. Then $x_2 = 2$, $x_3 = 0$ and $x_4 = 0$ by backtracking the optimal solutions.

4. (a) We consider this as a markov decision process where we start with the policy to use standard fuel and standard repair.

Define the states

$$s_k = \begin{cases} 1 & \text{if the car is working at the beginning of week } k \\ 0 & \text{if the car is in repair at the beginning of week } k \end{cases}$$

The state transition probabilities are

$$P = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} 0.2 & 0.8 \\ 0.05 & 0.95 \end{bmatrix}.$$

We need the expected costs for each step.

If the state $s_k = 0$, then the expected immediate cost is $C_0 = 50$ dollars. If the state $s_k = 1$, then the expected immediate cost is $C_1 = q_{10}p_{10} + q_{11}p_{11} = 250 * 0.05 + 10 * 0.95 = 22$ dollars.

Let $v_1 = 0$. The value determination equations

$$g + v_0 = C_0 + p_{00}v_0 + p_{01}v_1$$

$$g + v_1 = C_1 + p_{10}v_0 + p_{11}v_1$$

are

$$g + v_0 = 50 + (0.2v_0 + 0.8 * 0)$$

$$g + 0 = 22 + (0.05v_0 + 0.95 * 0)$$

gives $g = 402/17$, $v_0 = 560/17$ and $v_1 = 0$.

The total cost is now $50 * 10 * g = 500 * 402/17$.

(b) Now introduce the decision variables.

$$x_k = \begin{cases} 1 & \text{if standard fuel is used week } k \\ 2 & \text{if alternative fuel is used week } k \\ 3 & \text{if standard repair is used week } k \\ 4 & \text{if Heathcliff repair is used week } k \end{cases},$$

where the decisions $x_k = 1, 2$ are available for states $s_k = 1$ (working car) and the decisions $x_k = 3, 4$ are available for states $s_k = 0$ (car in repair).

From (a) $p_{00}(3) = p_{00} = 0.2$, $p_{01}(3) = p_{01} = 0.8$ and $p_{10}(1) = p_{10} = 0.05$, $p_{11}(1) = p_{11} = 0.95$.

Now $p_{00}(4) = 0.6$, $p_{01}(4) = 0.4$ and $p_{10}(2) = 0.1$, $p_{11}(2) = 0.9$.

To determine the immediate cost, note that the cost should only depend on the current state and decision in that state, it is necessary to take the one-time cost for the repair as the car is repaired, that is when the state transfers from 0 to 1.

Then

$$q_{10}(1) = 50, \quad q_{11}(1) = 10 \qquad \text{(st. fuel)}$$
$$q_{10}(2) = 50, \quad q_{11}(2) = 5 \qquad \text{(alt. fuel)}$$
$$q_{00}(3) = 50, \quad q_{01}(3) = 250 \qquad \text{(st. repair)}$$
$$q_{00}(4) = 50, \quad q_{01}(4) = 100 \qquad \text{(H. repair)}$$

Now

$$C_{11} = q_{10}(1)p_{10}(1) + q_{11}(1)p_{11}(1) = 50 * 0.05 + 10 * 0.99 = 12$$
$$C_{12} = q_{10}(2)p_{10}(2) + q_{11}(2)p_{11}(2) = 50 * 0.1 + 5 * 0.9 = 9.5$$
$$C_{03} = q_{00}(3)p_{00}(3) + q_{01}(3)p_{01}(3) = 50 * 0.2 + 250 * 0.8 = 210$$
$$C_{04} = q_{00}(4)p_{00}(4) + q_{01}(4)p_{01}(4) = 50 * 0.6 + 100 * 0.4 = 70$$

Starting policy (from (a)):
If $s_k = 0$, make decision $x_k = 3$.
If $s_k = 1$, make decision $x_k = 1$.
Solving the value determination equations again with new immediate costs. Let $v_1 = 0$.

$$g + v_0 = C_{03} + p_{00}(3)v_0 + p_{01}(3)v_1$$
$$g + v_1 = C_{11} + p_{10}(1)v_0 + p_{11}(1)v_1$$

are

$$g + v_0 = 210 + (0.2v_0 + 0.8 * 0)$$
$$g + 0 = 12 + (0.05v_0 + 0.95 * 0)$$

gives $g = 402/17$, $v_0 = 3960/17$ and $v_1 = 0$. Note that changing the time for the one-time cost do not change the expected cost per time step g.

To find out if it is optimal we do one step of the policy iteration.

For $i = 0$

$$\min_{k=3,4}\{C_{0k} + p_{00}(k)v_0 + p_{01}(k)v_1)\} =$$

$$= \min\{C_{03} + (p_{00}(3)v_0 + p_{01}(3)v_1), C_{04} + (p_{00}(4)v_0 + p_{01}(4)v_1)\}$$

$$= \min\{\underbrace{210 + (0.2 * 3960/17)}_{4362/17}, \underbrace{70 + (0.6 * 3960/17)}_{3566/17}\} = 3566/17 \text{ for } k = 4.$$

For $i = 1$

$$\min_{k=1,2}\{C_{1k} + (p_{10}(k)v_0 + p_{11}(k)v_1)\} =$$

$$= \min\{C_{11} + (p_{10}(1)v_1 + p_{11}(1)v_1), C_{12} + (p_{10}(2)v_1 + p_{11}(2)v_1)\}$$

$$= \min\{\underbrace{12 + (0.05 * 3960/17)}_{402/17}, \underbrace{9.5 + (0.1 * 3960/17)}_{1115/34}\} = 402/17 \text{ for } k = 1.$$

The starting policy is not optimal, to minimize the expected future cost he should change

Updated policy:
If $s_k = 0$, make decision $x_k = 4$.
If $s_k = 1$, make decision $x_k = 1$.

That is, use regular fuel and let Heathcliff repair the Mercedes.
Let $v_1 = 0$. The value determination equations

$$g + v_0 = C_{04} + p_{00}(4)v_0 + p_{01}(4)v_1$$

$$g + v_1 = C_{11} + p_{10}(1)v_0 + p_{11}(1)v_1$$

are

$$g + v_0 = 70 + (0.6v_0 + 0.4 * 0)$$

$$g + 0 = 12 + (0.05v_0 + 0.95 * 0)$$

gives $g = 166/9$, $v_0 = 1160/9$ and $v_1 = 0$.
The total cost is now $50 * 10 * g = 500 * 166/9$.

5. (a) This can be modelled as an economic order quantity model, EOQ model.
The demand is $d = 100$ kilos per day. The holding cost is $h = 0.05$ dollars per kilo and day. The ordering cost is $K = 100$ dollars.
Then the quantity to be ordered is

$$\hat{Q} = \sqrt{\frac{2dK}{h}} = 200\sqrt{10} \text{ kilos.}$$

He should order $200\sqrt{10}$ kilos each time and he should order every $2\sqrt{10}$ days.

(b) The total cost per cycle is $T_C = K + cQ + H$, where $Q = 100n$, n an integer and H is the holding cost $5n(n-1)/2$, *i.e.*,

$$K + c100n + 5n(n-1)/2$$

The length of a cycle is $Q/d = n$ days, so the total cost per time unit is then

$$T(n) = \frac{K}{n} + 100c + \frac{5(n-1)}{2}.$$

Now $\Delta T(n) = T(n+1) - T(n) = -\frac{K}{n(n+1)} + 5/2$, and

$$\Delta^2 T(n) = \Delta T(n+1) - \Delta T(n) = -K\left(\frac{1}{(n+1)(n+2)} - \frac{1}{n(n+1)}\right) =$$

$$= \frac{K}{n+1}\left(\frac{1}{n} - \frac{1}{n+2}\right) \geq 0,$$

which shows that T is integer-convex.
To find the optimal n note that $\Delta T(5) = -\frac{10}{3} + 5/2 \leq 0 \leq \Delta T(6) = -\frac{100}{42} + 5/2$, which tells us that $\hat{n} = 6$ is optimal.