KTH Mathematics

Suggested solutions for the exam in SF2863 Systems Engineering.
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1. Let n_i denote the level of contract i, where $i = 1, 2, 3.$.

Define functions f and g that we want to minimize, with the right properties. Frasse wants to maximize the lobby effects. Let $L_i(s_i)$ be defined by the table. Let $f(s_1, s_2, s_3) = -[L_1(s_1) + L_2(s_2) + L_3(s_3)]$ Clearly f is a separable function, and decreasing.

Integer-convexity follows by considering $\Delta f_i(1)$, i.e.

Δf	ΔL_1	ΔL_2	ΔL_3
$s_i = 0$	-10	-15	-15
$s_i = 1$	-5	-2	-3
$s_i = 2$	-3	-2	-2

which are increasing.

We want to minimize the cost, let $g(s_1, s_2, s_3) = c_1(s_1) + c_2(s_2) + c_3(s_3)$. Clearly g is a separable function, increasing, Integer-convexity follows by considering $\Delta c_i(1)$, i.e.

Δg	Δc_E	Δc_c	Δc_m
$s_i = 0$	6	7	8
$s_i = 1$	15	10	9
$s_i = 2$	17	11	11

which are increasing.

We can then apply the Marginal Allocation algorithm.

$-\Delta f / \Delta g$	$\frac{-\Delta f_1}{\Delta g_1}$	$\frac{-\Delta f_2}{\Delta g_2}$	$\frac{-\Delta f_3}{\Delta g_3}$
$s_i = 0$	$10/6 = 1.7$	$15/7 = 2.1$	$15/8 = 1.9$
$s_i = 1$	$5/15 = 0.33$	$2/10 = 0.2$	$3/9 = 0.33$
$s_i = 2$	$3/17 = 0.17$	$2/11 = 0.18$	$2/11 = 0.18$

The efficient allocations are therefore,
$S^{(0)} = (s_1 = 0, s_2 = 0, s_3 = 0), \quad f(s^{(0)}) = 0, \quad g(s^{(0)}) = 0$
$S^{(1)} = (s_1 = 0, s_2 = 1, s_3 = 0), \quad f(s^{(1)}) = 15, \quad g(s^{(1)}) = 7$
$S^{(2)} = (s_1 = 0, s_2 = 1, s_3 = 1), \quad f(s^{(2)}) = 30, \quad g(s^{(2)}) = 15$
$S^{(3)} = (s_1 = 1, s_2 = 1, s_3 = 1), \quad f(s^{(3)}) = 40, \quad g(s^{(3)}) = 21$
$S^{(4)} = (s_1 = 1, s_2 = 1, s_3 = 2), \quad f(s^{(3)}) = 43, \quad g(s^{(3)}) = 30$
$S^{(4')} = (s_1 = 2, s_2 = 1, s_3 = 1), \quad f(s^{(3)}) = 45, \quad g(s^{(3)}) = 36$
$S^{(5)} = (s_1 = 2, s_2 = 1, s_3 = 2), \quad f(s^{(3)}) = 48, \quad g(s^{(3)}) = 45$

Note that the order of the efficient solutions are not unique here, $S^{(4)}$ and $S^{(4')}$ can be taken in any order, *i.e.*, the fourth choice could be either lobby group 1 or 3 at contract level 2. However, if the budget constraint is 30 k\$, then the lobby group 3 at contract level 2 is the only choice.

2. The intensity of incoming calls (from outside) is $\lambda = 100$ per hour, and it is divided up to each station according to $\lambda_1 = 30$, $\lambda_2 = 50$, and $\lambda_3 = 20$.

Furthermore, including the feedback from station 1 the total arrival intensity at station 2 is $\tilde{\lambda}_2 = \lambda_2 + 0.2\lambda_1 = 56$ callers per hour. Similarly, including the feedback from station 2 the total arrival intensity at station 3 is $\tilde{\lambda}_3 = \lambda_3 + 0.5\tilde{\lambda}_2 = 48$ callers per hour.

The light traffic conditions have to be satisfied so therefore $\rho_1 = \lambda_1/(s_1\mu_1) < 1$ has to be satisfied, that is, s_1 has to be at least 2.

Similarly, $\rho_2 = \tilde{\lambda}_2/(s_2\mu_2) < 1$ has to be satisfied, that is, s_2 has to be at least 6.

Similarly, $\rho_3 = \tilde{\lambda}_3/(s_3\mu_3) < 1$ has to be satisfied, that is, s_3 has to be at least 3.

Use the formulas for $M|M|1$ systems to obtain $L_1 = \rho_1/(1-\rho_1) = 1$ caller, $L_2 = \rho_2/(1-\rho_2) = 14$ callers, $L_3 = \rho_3/(1-\rho_3) = 4$ callers. Then $L = L_1 + L_2 + L_3 = 19$ callers, and using the global version of Little's formula we have that the average time to pass the system is 19/100 hours + 30 seconds.

The difference in the modelling is that the service time for an individual caller is reduced by a factor s_i in the approximation. So if the queue is full then customers are served one-by-one at an increased speed, instead of in parallell at the normal speed. The average time to pass through the system is still similar. If the queue is not full then the caller still gets the fast service and then the average time to pass through the system is underestimated.

3. **Maximization formulation:**

Define the reduced problems

$$(\mathcal{P}_n) \quad \left[\begin{array}{ll} \max\limits_{x_n,\cdots,x_N} & \sum_{i=n}^N c_i x_i^2 \\ \text{s.t} & \sum_{i=n}^N x_i c_i \le C_n \\ & x_i \ge 0 \text{ for } i = n, \cdots, N. \end{array} \right]$$

where C_n = remaining budget for posters at locations $n, \cdots, N$.
The original problem is $(\mathcal{P}_1)$ starting at $C_1 = C$.

Let $f_n^*(C_n)$ = the optimal value of the optimization problem $(\mathcal{P}_n)$, and let $f_n^*(C_n, z)$ = the optimal value of the optimization problem $(\mathcal{P}_n)$ when z is the size of the poster at location n and and the rest of the budget is spent optimally.

Then

$$f_n^*(C_n) = \max_z \{ f_n^*(C_n, z) | \, C_n - c_n z \ge 0 \} = \max_z \left\{ z^2 I_n + f_n^*(C_n - z c_n) \big| \, C_n - z c_n \ge 0 \right\}.$$

We now solve it for $N = 3$, that is we need to solve for $f_n^*(C_n)$ for $C_n \le 10$ and $n = 2, 3$ to obtain $f_1^*(10)$.

Let $f_4^* \equiv 0$, then $f_3^*(C_3) = 4C_3^2$ if $x_3 = C_3$.

For $n = 2$,

$$f_2^*(C_2) = \max_z \left\{ 3z^2 + f_3^*(C_2 - 2z) \middle| C_2 - 2z \ge 0 \right\} = \max_z \left\{ 3z^2 + 4(C_2 - 2z)^2 \middle| C_2 - 2z \ge 0 \right\}.$$

Taking derivatives we see that $\hat{z} = 8/19C_2$. (Note that it satisfies the constraint) Then the maximum is attained either for $\hat{z} = 0$ or $\hat{z} = C_2/2$.

$f_2^*(C_2) = \max\{4C_2^2, 3/4C_2^2\} = 4C_2^2$ for $\hat{z} = 0$.

For $n = 1$,

$$f_1^*(10) = \min_z \left\{ z^2 + f_2^*(10 - z) \middle| 10 - z \ge 0 \right\} = \min_z \left\{ z^2 + 4(10 - z)^2 \middle| 10 - z \ge 0 \right\}.$$

Then the maximum is attained either for $\hat{z} = 0$ or $\hat{z} = 10$.

$f_1^*(10) = \max\{4 * 10^2, 10^2\} = 4 * 10^2$ for $\hat{z} = 0$. and the optimal poster sizes are given by

$x_1 = 0$, then $C_2 = 10$,
$x_2 = 0$, then $C_3 = C_2 - 2 * x_2 = 10$,
$x_3 = C_3 = 10$.

Minimization formulation:

Define the reduced problems

$$(\mathcal{P}_n) \quad \left[\begin{array}{ll} \min\limits_{x_n, \cdots, x_N} & \sum_{i=n}^N c_i x_i^2 \\ \text{s.t} & \sum_{i=n}^N x_i c_i = C_n \\ & x_i \ge 0 \text{ for } i = n, \cdots, N. \end{array} \right]$$

where C_n = remaining budget for posters at locations $n, \cdots, N$.
The original problem is $(\mathcal{P}_1)$ starting at $C_1 = C$.

Let $f_n^*(C_n)$ = the optimal value of the optimization problem $(\mathcal{P}_n)$, and let $f_n^*(C_n, z)$ = the optimal value of the optimization problem $(\mathcal{P}_n)$ when z is the size of the poster at location n and and the rest of the budget is spent optimally.

Then

$$f_n^*(C_n) = \min_z \left\{ f_n^*(C_n, z) \middle| C_n - c_n z \ge 0 \right\} = \min_z \left\{ z^2 I_n + f_n^*(C_n - z c_n) \middle| C_n - z c_n \ge 0 \right\}.$$

We now solve it for $N = 3$, that is we need to solve for $f_n^*(C_n)$ for $C_n \le 10$ and $n = 2, 3$ to obtain $f_1^*(10)$.

Let $f_4^* \equiv 0$, then $f_3^*(C_3) = 4C_3^2$ if $x_3 = C_3$.

For $n = 2$,

$$f_2^*(C_2) = \min_z \left\{ 3z^2 + f_3^*(C_2 - 2z) \middle| C_2 - 2z \ge 0 \right\} = \min_z \left\{ 3z^2 + 4(C_2 - 2z)^2 \middle| C_2 - 2z \ge 0 \right\}.$$

Taking derivatives we see that $\hat{z} = 8/19C_2$. (Note that it satisfies the constraint) Then

$f_2^*(C_2) = 12/19C_2^2$.

For $n = 1$,

$$f_1^*(10) = \min_z \left\{ z^2 + f_2^*(10-z) \middle| \, 10 - z \geq 0 \right\} = \min_z \left\{ z^2 + 12/19(10-z)^2 \middle| \, 10 - z \geq 0 \right\}.$$

Taking derivatives we see that $\hat{z} = 12/31 * 10$. (Note that it satisfies the constraint) Then $f_1^*(10) = 12/31 * 10^2$. and the optimal poster sizes are given by

$x_1 = 12/31 * 10$, then $C_2 = 10 - x_1 = 19/31 * 10$,
$x_2 = 8/19 * 19/31 = 8/31 * 10$, then $C_3 = C_2 - 2 * x_2 = 19/31 * 10 - 2 * 8/31 * 10 = 3/31 * 10$,
$x_3 = C_3 = 3 * 31 * 10$.

4. We consider this as a markov decision process where we start with the policy to never do any controversial statements.

Define the states

$$s_k = \begin{cases} 2 & \text{if H. is hot at the beginning of week } k \\ 1 & \text{if not at the beginning of week } k \end{cases}$$

Let the decision x_n at stage n be 1 if H. makes no controversial statement and 2 if he makes a controversial statement.

The state transition probabilities are (without making controversial statements)

$$P(1) = \begin{bmatrix} p_{00}(1) & p_{01}(1) \\ p_{10}(1) & p_{11}(1) \end{bmatrix} = \begin{bmatrix} 0.6 & 0.4 \\ 0.25 & 0.75 \end{bmatrix},$$

if he makes a controversial statement it is

$$P(2) = \begin{bmatrix} p_{00}(2) & p_{01}(2) \\ p_{10}(2) & p_{11}(2) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.6 & 0.4 \end{bmatrix}.$$

Then

$$q_{11}(1) = 50, \quad q_{12}(1) = 10 \qquad \text{(no statem.)}$$
$$q_{11}(2) = 50 + 20, \quad q_{12}(2) = 10 + 20 \qquad \text{(c.statem.)}$$
$$q_{21}(1) = 50, \quad q_{22}(1) = 10 \qquad \text{(no. statem.)}$$
$$q_{21}(2) = 50 - 40, \quad q_{22}(2) = 10 - 40 \qquad \text{(C. statem.)}$$

Now

$$C_{11} = q_{11}(1)p_{11}(1) + q_{12}(1)p_{12}(1) = 34$$
$$C_{21} = q_{21}(1)p_{21}(1) + q_{22}(1)p_{21}(1) = 20$$
$$C_{12} = q_{11}(2)p_{11}(2) + q_{12}(2)p_{12}(2) = 50$$
$$C_{22} = q_{21}(2)p_{21}(2) + q_{22}(2)p_{21}(2) = -6$$

Starting policy:
If $s_k = 1$, make decision $x_k = 1$.
If $s_k = 2$, make decision $x_k = 1$.

Solving the value determination equations again with new immediate costs. Let $v_2 = 0$.

$$g + v_1 = C_{11} + p_{11}(1)v_1 + p_{12}(1)v_2$$
$$g + v_2 = C_{21} + p_{21}(1)v_1 + p_{22}(1)v_2$$

are

$$g + v_1 = 34 + (0.6v_1 + 0.4 * 0)$$
$$g + 0 = 20 + (0.25v_1 + 0.75 * 0)$$

gives $g = 330/13$, $v_1 = 280/13$ and $v_2 = 0$.

To find out if it is optimal we do one step of the policy iteration.

For $i = 1$

$$\max_{k=1,2}\{C_{1k} + p_{11}(k)v_1 + p_{12}(k)v_2)\} =$$

$$= \max\{C_{11} + (p_{11}(1)v_1 + p_{12}(1)v_2), C_{12} + (p_{11}(2)v_1 + p_{12}(2)v_2)\}$$
$$= \max\{\underbrace{34 + (0.6 * 280/13)}_{610/13}, \underbrace{50 + (0.5 * 280/13)}_{790/13}\} = 790/13 \text{ for } k = 2.$$

For $i = 2$

$$\min_{k=1,2}\{C_{2k} + (p_{21}(k)v_1 + p_{22}(k)v_2)\} =$$

$$= \max\{C_{21} + (p_{21}(1)v_1 + p_{22}(1)v_2), C_{22} + (p_{21}(2)v_1 + p_{22}(2)v_2)\}$$
$$= \max\{\underbrace{50 + (0.25 * 280/13)}_{330/13}, \underbrace{-5 + (0.6 * 280/13)}_{90/13}\} =$$

for k=1.

The starting policy is not optimal, to maximize the expected future votes he should change

Updated policy:
If $s_k = 1$, make decision $x_k = 2$.
If $s_k = 2$, make decision $x_k = 1$.

That is, make controversial statements only when hot.

Let $v_2 = 0$. The value determination equations

$$g + v_1 = C_{12} + p_{11}(2)v_1 + p_{12}(2)v_2$$
$$g + v_2 = C_{21} + p_{21}(1)v_1 + p_{22}(1)v_2$$

are

$$g + v_1 = 50 + (0.5v_1 + 0.5 * 0)$$
$$g + 0 = 20 + (0.25v_1 + 0.75 * 0)$$

gives $g = 30$, $v_1 = 40$ and $v_2 = 0$.

The expected number of votes is now 30.

5. This can be modelled as a newvendor problem. Identify variables,

$$c = 3, \quad p = 10, \quad h = -1.$$

Then $F(\hat{k}) = \frac{p-c}{p+h} = \frac{10-3}{10-1} = 7/9.$.

Therefore, we want to find $\hat{k}$ such that

$$\sum_{k=\hat{k}}^{2N} \frac{2N-k}{L} = \sum_{k=\hat{k}}^{2N} \frac{2N-k}{N^2} = 2/9.$$

Let $x = 2N - \hat{k}$, then $x\frac{x}{2} = 2/9 * N^2$, *i.e.* $x = 2/3N$ and then the optimal number of T-shirts is $\hat{k} = 2 * N - x = 4/3N = 4/3 * 500.000$.