Optimization of spare part inventories on several levels

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This manuscript deals with some mathematical optimization models for multi-level inventories of expensive repairable items. Early models in this area were developed by Sherbrooke, and applied for the U.S. Air Force Logistics. Later, these early models have been extended in several directions, and used in a variety of civil application, see e.g. http://www.systecon.se/.

In the book "Optimal inventory modeling of systems: multi-echelon techniques", the author Craig Sheerbrooke, who has developed several important models in this area, describes a part of the model considered in this manuscript as follows (in "aircraft language"): "When a malfunction is diagnosed on an aircraft, the malfunctioning item is removed from the aircraft and brought into base supply. If a spare is available, it is issued and installed on the aircraft; otherwise a backorder is established ... which implies that there is a "hole" in an aircraft that causes it to be grounded ...".

Further quotations from his description will follow later.

We make frequent reference to the companion manuscript *On marginal allocation*, abbreviated MALLOC, which we assume that the reader has access to.

1 Model 1 (one base, one LRU)

We begin by considering the simplest model, which is characterized as follows:

1. There is only one *base*, with its own inventory of spare items and its own *workshop.*

2. There is only one organizational level. (The case with a central *depot* is considered later.)

3. Only one type of items is considered, to make it less abstract it is here referred to as *aircraft engine*. The considered item "aircraft engine" is an example of a so called "line replaceable unit", abbreviated LRU.

We assume that the LRU:s can be in two states, either *functioning* or *defect*, and that when a LRU is defect it has to be repaired in the workshop to become functioning again.

The rate at which aircrafts with a defect engine arrive at the base is modelled by a Poisson process with intensity λ engines per time unit.

When a defect engine has arrived, it is immediately removed from the aircraft and brought into the workshop. If the inventory of functioning engines is non-empty, such an engine is immediately installed into the aircraft which is then operable again. But if the inventory of functioning engines is empty, a *backorder* is established and the aircraft is grounded and out of order for the time being.

When a defect engine has been repaired in the workshop, it is immediately put in the inventory of spare engines. The repair times for defect engines are assumed to be independent and equally distributed random variables with expected value T time units. (This implicitly assumes that the workshop has the capacity to repair any number of engines in parallell at constant service rate) According to a theorem by Palm (see Appendix), this implies that the number of engines in the workshop, at a randomly chosen time, is a Poisson distributed random variable with expected value λT.

An important decision variable in the model is the following:

$s =$ the number of spare engines which has been purchased for the base, *i.e.*, the number of engines in the inventory when there is no engine in the workshop.

The integer s is referred to as the "stock level".

If the system is considered at a given (randomly chosen) time, one has the following natural random variables:

X = the number of engines currently in the workshop.
OH = the number of engines currently available in the inventory (*on hand*).
BO = the number of currently grounded aircrafts waiting for an engine (*backorders*).

Between these random variables, which can only take on non-negative integer values, the following relation holds:

$$BO - OH = X - s\,. \tag{1.1}$$

Moreover, at each time at least one of BO and OH is zero.
Therefore, BO and OH can be expressed as the following functions of X and s:

$$BO = (X - s)^+ = \max\{\,0\,,\ X - s\} \ \text{ and } \ OH = (s - X)^+ = \max\{\,0\,,\ s - X\}. \tag{1.2}$$

1.1 Expected number of backorders in Model 1

As mentioned above, X is a Poisson distributed random variable with expected value λT, *i.e.*, the probability mass function is

$$p(k) := P(X = k) = \frac{(\lambda T)^k}{k!}\, e^{-\lambda T} \text{ for } k = 0, 1, 2, \cdots\,. \tag{1.3}$$

A key quantity is the expected value of the number of back orders, *i.e.*, the average number of aircrafts that are grounded while waiting for a working engine. This quantity can be expressed as $\mathrm{E}[BO] = \mathrm{E}[(X-s)^+]$, which will from now on be denoted $\mathrm{EBO}(s)$. Thus,

$$\mathrm{EBO}(s) = \mathrm{E}[BO] = \mathrm{E}[(X - s)^+]. \tag{1.4}$$

Since the probability distribution of X is given by (1.3), the computation of $\mathrm{EBO}(s)$ can be done recursively as follows.

First, $p(k)$ can be computed recursively, since $p(0) = e^{-\lambda T}$ and

$$p(k+1) = \frac{\lambda T}{k+1}\, p(k), \quad \text{for } k = 0, 1, 2, \ldots \tag{1.5}$$

Next, let $R(s)$ = the probability for shortage, i.e.

$$R(s) = P(X > s) = \sum_{k=s+1}^{\infty} p(k) \quad \text{for } s = 0, 1, 2, \ldots \tag{1.6}$$

$R(s)$ can also be computed recursively, since $R(0) = 1 - p(0)$ and

$$R(s+1) = R(s) - p(s+1), \quad \text{for } s = 0, 1, 2, \ldots \tag{1.7}$$

Further,

$$\mathrm{EBO}(0) = \mathrm{E}[(X-0)^+] = \mathrm{E}[X] = \lambda T\,, \tag{1.8}$$

while

$$\mathrm{EBO}(s) = \mathrm{E}[(X-s)^+] = \sum_{k=s+1}^{\infty} (k-s)p(k)\,, \tag{1.9}$$

and

$$\mathrm{EBO}(s+1) = \sum_{k=s+2}^{\infty} (k-s-1)p(k) = \sum_{k=s+1}^{\infty} (k-s-1)p(k)\,. \tag{1.10}$$

From (1.6), (1.9) and (1.10) the following simple recursion formula is obtained,

$$\mathrm{EBO}(s+1) = \mathrm{EBO}(s) - R(s), \quad \text{för } s = 0, 1, 2, \dots \tag{1.11}$$

Assume that $\mathrm{EBO}(s)$ should be computed for $s = 0, 1, \dots, s^{\max}$. This can easily be done using the following Matlab statements. (As the indexing of vectors in Matlab starts with 1, $p(0)$ above will be called p(1) in Matlab, etc.)

```
lamT = lambda*T;
p(1) = exp(-lamT);
R(1) = 1 - p(1);
EBO(1) = lamT;
for k=1:smax
  k1=k+1;
  p(k1) = lamT*p(k)/k;
  R(k1) = R(k) - p(k1);
  EBO(k1) = EBO(k) - R(k);
end
```

Note that since $p(s) > 0$ for all $s \geq 0$, it follows that $R(s+1) < R(s)$. Moreover,

$$\Delta\mathrm{EBO}(s) := \mathrm{EBO}(s+1) - \mathrm{EBO}(s) = -R(s) < 0\,, \text{ and} \tag{1.12}$$

$$\Delta^2\mathrm{EBO}(s) := \Delta\mathrm{EBO}(s+1) - \Delta\mathrm{EBO}(s) = p(s+1) > 0\,, \tag{1.13}$$

which means that $\mathrm{EBO}(s)$ is decreasing and *integer-convex*, see MALLOC.

If necessary, one can also compute the variance of the number of back orders, *i.e.*,

$$\mathrm{VBO}(s) := \mathrm{Var}[(X-s)^+] = \mathrm{E}[((X-s)^+)^2] - (\mathrm{E}[(X-s)^+])^2, \tag{1.14}$$

as follows. First let

$$\mathrm{EBO2}(s) = \mathrm{E}[((X-s)^+)^2] = \sum_{k=s+1}^{\infty} (k-s)^2 p(k). \tag{1.15}$$

Then

$$\mathrm{EBO2}(s+1) = \sum_{k=s+2}^{\infty} (k-s-1)^2 p(k) = \sum_{k=s+1}^{\infty} (k-s-1)^2 p(k), \tag{1.16}$$

from which it is obtained, after some manipulations, that

$$\mathrm{EBO2}(s) - \mathrm{EBO2}(s+1) = \mathrm{EBO}(s) + \mathrm{EBO}(s+1). \tag{1.17}$$

It follows, after some additional manipulations, that

$$\mathrm{VBO}(s+1) = \mathrm{VBO}(s) - (\mathrm{EBO}(s) + \mathrm{EBO}(s+1))(1 - R(s)), \tag{1.18}$$

with the initial value

$$\mathrm{VBO}(0) = \mathrm{Var}[X] = \lambda T. \tag{1.19}$$

Hence, only two additional statements in Matlab are needed:

```
VBO(1) = lamT;
VBO(k1) = VBO(k) - (EBO(k) + EBO(k1))(1 - R(k));
```

Exercise: Verify (1.17) and (1.18).

1.2 An optimization problem under Model 1

We now consider the following possible optimization problem under Model 1:

$$\text{minimize} \;\; f(s) = q\,\mathrm{EBO}(s) + cs\,, \;\; \text{subject to } s \in \{0, 1, 2, 3, \ldots\}\,. \tag{1.20}$$

where the constant $c > 0$ can be interpreted as cost per spare engine, while the constant $q > 0$ can be interpreted as the cost per grounded (out of order) aircraft.

Let $\Delta f(s) = f(s+1) - f(s)$. Then

$$\Delta f(s) = q\,\Delta\mathrm{EBO}(s) + c = -qR(s) + c\,. \tag{1.21}$$

Since $q > 0$ and $\mathrm{EBO}(s)$ is integer-convex, it follows that $f(s)$ is integer-convex, and then the following proposition follows from Prop 1.1 in MALLOC.

Proposition 1.1: Let $f(s) = q\,\mathrm{EBO}(s) + cs$, for $s \in \{0, 1, 2, 3, \ldots\}$. Then

$$\hat{s} = 0 \text{ minimizes } f(s) \text{ if and only if } R(0) \leq \frac{c}{q}\,, \tag{1.22}$$

$$\hat{s} > 0 \text{ minimizes } f(s) \text{ if and only if } R(\hat{s}) \leq \frac{c}{q} \leq R(\hat{s} - 1)\,. \tag{1.23}$$

A simple algorithm for solving problem (1.20) is then to calculate $R(s)$ for $s = 0, 1, 2, \cdots$ until an $\hat{s}$ is found such that (for the first time) $R(\hat{s}) \leq c/q$. Then $\hat{s}$ is an optimal solution.

2 Model 2 (one base, several LRU)

In this model, we extend Model 1 to the case that there are several different line replaceable units (LRU) in each aircraft. More precisely, we assume that there are $n > 1$ different LRU, here referred to as $\mathrm{LRU}_1, \ldots, \mathrm{LRU}_n$. In the aircraft example considered before, we can in addition of engines also consider transducers, actuators and electronic modules. As soon as any of these is defect, it must be replaced by a functioning one before the aircraft can be used again. The assumptions and notations from Model 1 (which corresponds to $n = 1$) are then generalized as follows.

Aircrafts with defect LRU_j arrive to the base according to a Poisson process with intensity λ_j. It is assumed that the LRU:s of different type malfunction independently of each others. The repair times for LRU_j are assumed to be independent and equally distributed stochastic variables with expected value T_j.

The important decision variables are now $s_1, \cdots, s_n$, where

s_j = the number of spare units of LRU_j which have been purchased for the base, *i.e.*, the number of LRU_j in the inventory when there is no LRU_j in the workshop.

The integer s_j denote the stock level of LRU_j, and introduce the vector $\mathbf{s}$ of stock levels $\mathbf{s} := (s_1, \ldots, s_n)^\mathsf{T}$.

Let c_j be the cost per spare unit of LRU_j, and introduce the vector $\mathbf{c} = (c_1, \ldots, c_n)^\mathsf{T}$.

Then $C(\mathbf{s}) := \mathbf{c}^\mathsf{T}\mathbf{s}$ = the total cost of spare items at the base.

Consider the system at a given (randomly chosen) time, and let

X_j = the number of LRU_j in the workshop.

According to Palm's theorem, X_j is Poisson distributed with expected value $\lambda_j T_j$, *i.e.*,

$$p_j(k) = P(X_j = k) = \frac{(\lambda_j T_j)^k}{k!}\, e^{-\lambda_j T_j}. \tag{2.1}$$

Let $\mathrm{EBO}(\mathbf{s})$ be the average number of aircrafts grounded due to shortage of some item. Then (since the defects occur independently)

$$\mathrm{EBO}(\mathbf{s}) = \sum_{j=1}^{n} \mathrm{EBO}_j(s_j) = \sum_{j=1}^{n} \mathrm{E}[(X_j - s_j)^+], \tag{2.2}$$

where $\mathrm{EBO}_j(s_j)$ is the average number of aircrafts grounded due to shortage of LRU_j.

As in Model 1, it holds that

$$\mathrm{EBO}_j(s_j + 1) = \mathrm{EBO}_j(s_j) - R_j(s_j)\,, \tag{2.3}$$

where

$$R_j(s_j) = P(X_j > s_j) = \sum_{k=s_j+1}^{\infty} p_j(k)\,. \tag{2.4}$$

Consequently, we obtain recursive equations of the same type as in Model 1.

Now let $S = \{\mathbf{s} = (s_1, \ldots, s_n)^\mathsf{T} \mid s_j \in \{0, 1, 2, \ldots\} \text{ for all } j\}$.

S is an infinite set. In practice, the set S can be made finite by only considering the points in S that satisfy $C(\mathbf{s}) \leq C^{\max}$, where $C^{\max}$ is a upper limit for how much the spare-parts can possibly be allowed to cost. However, the number of elements in S is typically extremely large, for realistic values of n and $C^{\max}$.

2.1 Efficient solutions of Model 2

Each $\mathbf{s} \in S$ induces a spare parts cost $C(\mathbf{s})$ and an average number of back orders $\mathrm{EBO}(\mathbf{s})$. The vector $\hat{\mathbf{s}} \in S$ is an *efficient solution* and $(C(\hat{\mathbf{s}}), \mathrm{EBO}(\hat{\mathbf{s}}))$ is an *efficient point* if there is a constant $q > 0$ such that $\hat{\mathbf{s}}$ is an optimal solution to the following optimization problem in $\mathbf{s}$:

$$\min\ C(\mathbf{s}) + q\,\mathrm{EBO}(\mathbf{s}),\ \text{subject to } \mathbf{s} \in S. \tag{2.5}$$

The following geometrical interpretation of the efficients points was provided in MALLOC: Let $M = \{(C(\mathbf{s}), \mathrm{EBO}(\mathbf{s})) \mid \mathbf{s} \in S\}$ and assume that all the points in M are plotted in a coordinate system where the horizontal axis shows $C(\mathbf{s})$ and the vertical axis shows $\mathrm{EBO}(\mathbf{s})$.

The *convex hull* of M is defined as the smallest convex set that contains the whole set M. The *efficient curve* corresponding to the set M is defined as the piecewise linear curve that constitutes the "southwestern boundary" of the convex hull of M.
The points $(C(\mathbf{s}), \mathrm{EBO}(\mathbf{s})) \in M$ which lie on this curve are called *efficient points*, and the corresponding vectors $\mathbf{s}$ are called *efficient solutions*.

Typically, we are interested in determining the efficient curve for a given situation, with given values on the above parameters. It turns out that even if the numbers of elements in S and M are extremely large, it is surprisingly easy to determine the efficient curve! We describe below how this is done.

Proposition 2.1: $\hat{\mathbf{s}} \in S$ minimizes $C(\mathbf{s}) + q\,\mathrm{EBO}(\mathbf{s})$ on S if and only if the following conditions are satisfied for each $j = 1, \ldots, n$:

$$\frac{R_j(0)}{c_j} \leq \frac{1}{q} \quad \text{if} \ \ \hat{s}_j = 0\,, \tag{2.6}$$

$$\frac{R_j(\hat{s}_j)}{c_j} \leq \frac{1}{q} \leq \frac{R_j(\hat{s}_j - 1)}{c_j} \quad \text{if} \ \ \hat{s}_j > 0\,. \tag{2.7}$$

Proposition 2.2: $\hat{\mathbf{s}} \in S$ is an efficient solution and $(C(\hat{\mathbf{s}}), \text{EBO}(\hat{\mathbf{s}})) \in M$ is an efficient point if and only if there is a $q > 0$ such that the conditions (2.6)–(2.7) are satisfied for each $j = 1, \ldots, n$.

Proposition 2.3: Assume that $\hat{\mathbf{s}} \in S$ is an efficient solution and let $\hat{C} = C(\hat{\mathbf{s}})$ and $\widehat{\text{EBO}} = \text{EBO}(\hat{\mathbf{s}})$.
Then $\hat{\mathbf{s}}$ is an optimal solution to both the following optimization problems:

$$\min C(\mathbf{s}),\ \text{subject to } \text{EBO}(\mathbf{s}) \leq \widehat{\text{EBO}}, \mathbf{s} \in S \tag{2.8}$$

$$\min \text{EBO}(\mathbf{s}),\ \text{subject to } C(\mathbf{s}) \leq \hat{\text{C}}, \mathbf{s} \in S \tag{2.9}$$

2.2 Marginal Allocation Algorithm for Model 2

We now describe a surprisingly simple algorithm for determining the efficient curve. The algorithm generates efficient solutions $\mathbf{s}^{(1)}, \mathbf{s}^{(2)}, \mathbf{s}^{(3)}, \ldots$ " from left to right", *i.e.*, each new generated point has a higher value on $C(\mathbf{s})$ but a lower value on $\text{EBO}(\mathbf{s})$ than the previously generated point. Throughout the algorithm $\mathbf{s}^{(i)}$ denotes the i:th generated efficient solution, $C^{(i)}$ denotes the corresponding spare-part cost $C(\mathbf{s}^{(i)})$, and $\text{EBO}^{(i)}$ denotes the corresponding expected number of backorders $\text{EBO}(\mathbf{s}^{(i)})$. The algorithm terminates when there is no longer any efficient solution with $C(\mathbf{s}) \leq C^{\max}$.

Step 0:
Generate a table with n columns as follows. For $j = 1, \ldots, n$, fill the j:th column from the top and down with the quotients $R_j(0)/c_j$, $R_j(1)/c_j$, $R_j(2)/c_j$, etc.. (A moderately large number of quotients will suffice, since additional quotients can be calculated later if it should be proven necessary.) Note that the quotients are positive and strictly decreasing in each column.
Set $i = 0$, $s_1 = \ldots = s_n = 0$, $\mathbf{s}^{(0)} = (0, \ldots, 0)^{\mathsf{T}}$, $C^{(0)} = 0$ and $\text{EBO}^{(0)} = \sum_{j=1}^{n} \lambda_j T_j$.
Let all the quotients in the table be *uncanceled*.

Step 1:
Select the largest uncanceled quotient in the table (if there are several equally large, choose one of these arbitrarily). Cancel this quotient and let k be the number of the column from which the quotient was canceled.

Step 2:
Let $i := i + 1$. Then let $s_k^{(i)} := s_k^{(i-1)} + 1$ and $s_j^{(i)} := s_j^{(i-1)}$ for all $j \neq k$.

Set $C^{(i)} = C^{(i-1)} + c_k$, $\text{EBO}^{(i)} = \text{EBO}^{(i-1)} - R_k(s_k(i-1))$,
If $C^{(i)} \geq C^{\max}$, terminate the algorithm. Otherwise, go back to Step 1.

2.3 Some properties of the algorithm

Note that each generated solution $\mathbf{s}^{(i)}$ differs from the previously generated solution $\mathbf{s}^{(i-1)}$ in just one component. The name of the algorithm stems from the fact that

$$\frac{R_j(s_j)}{c_j} = -\frac{\Delta\mathrm{EBO}_j(s_j)}{c_j} = \frac{\text{decrease in EBO}(\mathbf{s})}{\text{increase in } C(\mathbf{s})} \text{ if } s_j \text{ is increased by 1.}$$

Hence, in each step of the algorithm, one increases the s_j which gives marginally the largest reduction of EBO($\mathbf{s}$) per invested crown/euro/dollar.

The following two propositions are immediate consequences of Prop 3.1 and 3.2 in MALLOC:

Proposition 2.3: Each generated solution $\mathbf{s}^{(i)}$ is an efficient solution.

Proposition 2.4: If all quotients $R_j(s_j)/c_j$ in the original table are different, then the algorithm generates all efficient solutions that satisfy $C(\mathbf{s}) \leq C^{\max}$.

These conditions determine $\hat{\mathbf{s}}$ uniquely. However, this solution will actually be generated by the algorithm in the stage where the latest canceled quotient is $> 1/q$, while the largest quotient that has not yet been canceled is $< 1/q$.

Proposition 2.5: Assume that $\hat{\mathbf{s}} \in S$ is an efficient solution and let $\widehat{C} = C(\hat{\mathbf{s}})$ and $\widehat{\mathrm{EBO}} = \mathrm{EBO}(\hat{\mathbf{s}})$. Then $\hat{\mathbf{s}}$ is an optimal solution to both the following optimization problems:

$$\text{minimize } C(\mathbf{s}) \text{ subject to } \mathrm{EBO}(\mathbf{s}) \leq \widehat{\mathrm{EBO}}\,,\ \mathbf{s} \in S\,. \tag{2.10}$$

$$\text{minimize } \mathrm{EBO}(\mathbf{s}) \text{ subject to } C(\mathbf{s}) \leq \widehat{C}\,,\ \mathbf{s} \in S\,. \tag{2.11}$$

We say that $\mathbf{s} \in S$ is "strictly better" than $\hat{\mathbf{s}} \in S$ if $C(\mathbf{s}) < C(\hat{\mathbf{s}})$ and $\mathrm{EBO}(\mathbf{s}) \leq \mathrm{EBO}(\hat{\mathbf{s}})$, or if $\mathrm{EBO}(\mathbf{s}) < \mathrm{EBO}(\hat{\mathbf{s}})$ and $C(\mathbf{s}) \leq C(\hat{\mathbf{s}})$. After adopting this notation, the last proposition above states that if $\hat{\mathbf{s}} \in S$ is an efficient solution, then no other $\mathbf{s} \in S$ is strictly better than $\hat{\mathbf{s}}$. (The reverse, however, is not true. There may exist $\bar{\mathbf{s}} \in S$ which are not efficient solutions, but nevertheless satisfy that no $\mathbf{s} \in S$ is strictly better than $\bar{\mathbf{s}}$.)

3 Model 3 (a central depot and several bases, one LRU)

In this METRIC model (Multi-Echelon Technique for Recoverable Item Control) there are two organizational levels, but (to begin with) only one type of LRU, which is again referred to as *aircraft engine*. On the lowest organizational level there are n “ bases”, each one equipped with a *local inventory* of spare engines, but no workshop. On the highest level there is a *central depot* with a central workshop and a *central inventory* of spare engines.

Decision variables in the model are:
s_j = the number of spare engines at base j, for $j = 1, \ldots, n$, and
s_0 = the number of spare engines at the depot.

At base j errors, *i.e.*, aircrafts with defect engines, arrive according to a Poisson process with intensity λ_j. When a defect engine arrives at the base it is immediately replaced by a functioning engine from the local inventory, unless the local inventory is empty.

If there is no engine in the local inventory that can replace the defect engine a back order is established at the base, and the corresponding aircraft is grounded.

The defect engine is sent directly to the central workshop. At the same time, a functioning engine is sent from the central inventory to the local inventory at the base. If the central inventory is empty, so no functioning engine can be sent to the base, a *depot backorder* is established. This does not necessarily imply that an aircraft is grounded, but the risk of back orders at the bases increases.

The time it takes to transport a defect engine from a base to the depot, T_{bd}, is assumed to be deterministic and known, and the same is assumed for the time it takes to transport a functioning engine from the depot to a base, T_{db}. For simplicity, we assume that there is no difference between the bases in this respect.

The repair time for a defect engine at the central workshop is assumed to be a random variable with expected value T_{rep}. An important assumption (approximation) in the model is that these repair times are independent and equally distributed.

The question now is how large the inventories of spare engines should be, both locally at the bases and centrally at the depot. In particular, we are interested in determining the efficient curve which relates the cost of spare engines (horizontal axis) to the average number of grounded aircrafts (vertical axis) when the purchased spare engines are allocated in an optimal way.

3.1 Analysis of the situation at the depot

Let X_0 = the number of defect engines that are in, or on their way to the workshop.

From the given conditions, it follows that defect engines arrive to the workshop according to a Poisson process with intensity $\lambda_0 = \lambda_1 + \ldots + \lambda_J$. As the repair times have been assumed independent, it follows from Palm's theorem that X_0 is a Poisson distributed random variable with

$$\mathrm{E}[X_0] = \lambda_0 T_0, \text{ where } T_0 = T_{bd} + T_{rep}. \tag{3.1}$$

This implies that it is possible to compute $\mathrm{EBO}_0(s_0) = \mathrm{E}[(X_0 - s_0)^+]$, *i.e.*, the average number of back orders at the depot, with the same type of recursive equations that was used in Model 1. If needed, one can also compute $\mathrm{VBO}_0(s_0) = \mathrm{Var}[(X_0 - s_0)^+]$.

3.2 Analysis of the situation at a base

Let $X_j =$ the number of engines in the *pipeline* at base j, *i.e.*, the number of defect engines that have been sent from base j to the central workshop, but for which replacement engines have still not been delivered to the local supply at base j.

It holds that $X_j = Y_j + Z_j$, where

$Y_j =$ the number of defect engines that have been received at base j in the last T_{db} time units,

$Z_j =$ the number of defect engines that were received at base j more than T_{db} time units ago, which where depot backorders T_{db} time units ago.

Since Y_j is the number of Poisson arrivals in a given time interval, Y_j is a Poisson distributed random variable with $\mathrm{E}[Y_j] = \lambda_j T_{db}$. For a Poisson distributed random variable the variance is equal to the expected value. Thus, $\mathrm{Var}[Y_j] = \lambda_j T_{db}$.

Let $Z_0 = Z_1 + \ldots + Z_n =$ the total number of back orders at the depot T_{db} time units ago. Note that Z_0 has the same distribution as $(X_0 - s_0)^+$.

Since Z_j is the part of Z_0 that corresponds to base j, and since, on average, λ_j/λ_0 of all the defect engines at the central workshop originate from base j, we obtain

$$\mathrm{E}[Z_j] = \frac{\lambda_j}{\lambda_0}\mathrm{E}[Z_0] = \frac{\lambda_j}{\lambda_0}\mathrm{EBO}_0(s_0). \tag{3.2}$$

A formal derivation of this expression is described now. For a given $Z_0 = z_0$, Z_j is binomial distributed, $\mathrm{Bin}(n, p_j)$, with parameters $n = z_0$ and $p_j = \dfrac{\lambda_j}{\lambda_0}$. This gives us the following conditional expected values and variances:

$$\mathrm{E}[Z_j|Z_0] = p_j Z_0 \quad \text{och} \quad \mathrm{Var}[Z_j|Z_0] = p_j(1 - p_j)Z_0\,. \tag{3.3}$$

Using well-known computational rules for conditional expected values and conditional variances (see e.g. BETA) we get

$$\mathrm{E}[Z_j] = \mathrm{E}[\mathrm{E}[Z_j|Z_0]] \quad \text{och} \quad \mathrm{Var}[Z_j] = \mathrm{E}[\mathrm{Var}[Z_j|Z_0]] + \mathrm{Var}[\mathrm{E}[Z_j|Z_0]], \tag{3.4}$$

which leads to the following expressions:

$$\mathrm{E}[Z_j] = p_j \mathrm{E}[Z_0] = p_j \mathrm{E}[(X_0 - s_0)^+] = \frac{\lambda_j}{\lambda_0} \mathrm{EBO}_0(s_0), \tag{3.5}$$

$$\mathrm{Var}[Z_j] = p_j(1 - p_j)\mathrm{EBO}_0(s_0) + p_j^2 \mathrm{VBO}_0(s_0), \tag{3.6}$$

where $\mathrm{VBO}_0(s_0) = \mathrm{Var}[(X_0 - s_0)^+]$ = the variance of the number of back orders at the depot. A reformulation gives

$$\mathrm{Var}[Z_j] = \mathrm{E}[Z_j] + \frac{\lambda_j^2}{\lambda_0^2}(\mathrm{VBO}_0(s_0) - \mathrm{EBO}_0(s_0)). \tag{3.7}$$

As Y_j and Z_j are independent (the arrival of defect units after a given time instant is independent of the arrivals occurring before that time instant) we finally get the following expressions for the expected value and variance of the number of engines in the pipeline:

$$\mathrm{E}[X_j] = \mathrm{E}[Y_j + Z_j] = \lambda_j (T_{db} + \frac{\mathrm{EBO}_0(s_0)}{\lambda_0}), \tag{3.8}$$

$$\mathrm{Var}[X_j] = \mathrm{E}[X_j] + \frac{\lambda_j^2}{\lambda_0^2}(\mathrm{VBO}_0(s_0) - \mathrm{EBO}_0(s_0)). \tag{3.9}$$

In the METRIC model one makes the assumption that the pipeline times of the engines at a base are *independent* and equally distributed random variables. Here the pipeline time for an engine is defined as the time *from* which the defect engine arrives at a base and is sent to the workshop *until* the local inventory at the base has received a corresponding functioning engine in exchange.

From Palm's theorem it then follows that X_j is a Poisson distributed random variable with expected value given by (3.8) above, *i.e.*,

$$p_j(k) = P(X_j = k) = \frac{(\lambda_j T_j)^k}{k!} e^{-\lambda_j T_j}, \text{ where } T_j = T_{db} + \frac{\mathrm{EBO}_0(s_0)}{\lambda_0}. \tag{3.10}$$

This means that it is possible to compute $\mathrm{EBO}_j(s_j) = \mathrm{E}[(X_j - s_j)^+]$, *i.e.*, the average number of back orders at the base j, with the same type of recursive equations that was used in Model 1. However, note that $\mathrm{EBO}_j(s_j)$ also depends on s_0, since $\mathrm{E}[X_j]$ depends on s_0. We will therefore use the notation $\mathrm{EBO}_j(s_0, s_j)$.

If the assumption in METRIC was correct, the variance and the expected value of X_j would be the same. However, this is not the case when $s_0 > 0$, since

$$\mathrm{VBO}_0(s_0) > \mathrm{EBO}_0(s_0), \quad \text{for } s_0 > 0, \tag{3.11}$$

and as a consequence, according to (3.9),

$$\mathrm{Var}[X_j] > \mathrm{E}[X_j], \quad \text{for } s_0 > 0. \tag{3.12}$$

Yet, according to the above, if $s_0 = 0$ it holds that

$$\mathrm{Var}[X_j] = \mathrm{E}[X_j] = \lambda_j(T_{db} + T_{rep} + T_{bd}), \quad \text{for } s_0 = 0. \tag{3.13}$$

In the VARI-METRIC model one instead assumes that if $s_0 > 0$, then X_j is a translated negative binomial distributed random variable with expected value given by (3.8) *and* variance given by (3.9) above. More precisely, it is assumed that for $s_0 > 0$

$$P(X_j = 0) = \left(\frac{1}{V_j}\right)^{\frac{\mu_j}{V_j - 1}}, \quad \text{and} \tag{3.14}$$

$$P(X_j = k+1) = \frac{\mu_j + (V_j - 1)k}{V_j(k+1)} \cdot P(X_j = k), \quad k = 0, 1, 2, \ldots \tag{3.15}$$

where $\mu_j = \mathrm{E}[X_j]$ and $V_j = \dfrac{\mathrm{Var}[X_j]}{\mathrm{E}[X_j]}$.

For $s_0 = 0$ it is still assumed that X_j is a Poisson distributed random variable with expected value and variance given by (3.13) above.

It is now possible to compute $\mathrm{EBO}_j(s_0, s_j) = \mathrm{E}[(X_j - s_j)^+]$, *i.e.*, the average number of back orders at base j, with almost the same type of recursive equations as in the METRIC model. The only difference is that now (3.14) and (3.15) are used in the computation of the probabilities for the number of units in the pipeline (if $s_0 > 0$).

It can easily be verified that if $V_j \longrightarrow 1$ then (3.14) and (3.15) become

$$P(X_j = 0) = e^{-\mu_j}, \quad \text{and} \tag{3.16}$$

$$P(X_j = k+1) = \frac{\mu_j}{(k+1)} \cdot P(X_j = k), \quad k = 0, 1, 2, \ldots \tag{3.17}$$

as for the Poisson distribution.

3.3 Efficient curve for Model 3

Let $\mathbf{s} = (s_1, \ldots, s_n)^\mathsf{T}$, and let

$$\mathrm{EBO}(s_0, \mathbf{s}) = \sum_{j=1}^{n} \mathrm{EBO}_j(s_0, s_j)\,, \tag{3.18}$$

i.e., $\mathrm{EBO}(s_0, \mathbf{s})$ = the average number of aircrafts that are grounded at a base.

Let c_0 = the cost per spare engine, and let $C(s_0, \mathbf{s})$ = the total cost of the spare engines, *i.e.*,

$$C(s_0, s) = c_0 s_0 + \sum_{j=1}^{n} c_0 s_j. \tag{3.19}$$

It will now be described how to determine the efficient curve in a coordinate system where the horizontal axis shows $C(s_0, s)$ and the vertical axis shows $\mathrm{EBO}(s_0, \mathbf{s})$. Only efficient solutions with $C(s_0, \mathbf{s}) \le C^{\max}$ will be considered, where $C^{\max}$ is a upper bound on the possible cost for spare engines. Equivalently, this can be expressed as a bound on the number of spare engines as $s_0 + \sum_{j=1}^{n} s_j \le s^{\max}$, where $s^{\max}$ is the largest integer such that $cs^{\max} \le C^{\max}$.

3.4 Algorithm for a fixed s_0

In this section it is assumed that s_0 is held fixed (to a non-negative integer).

Then it is possible, by using the marginal allocation algorithm of Model 2, to determine the efficient solutions for allocation of spare engines to the bases. More precisely, first $\mathrm{EBO}_0(s_0)$ is calculated and then the algorithm in Section 2.2 is applied with the following modifications:
- the index j now corresponds to base number j (and not LRU_j),
- the cost coefficients c_j are now all equal to c (the cost for a spare engine)
- the time constants T_j are now all equal to $T_{db} + \mathrm{EBO}_0(s_0)/\lambda_0$ (the expected pipeline times)

The results from the algorithm will be a set of efficient points $\mathbf{s}^{(k)}$, the corresponding expected number of grounded airscrafts,

$$\mathrm{EBO}(s_0, \mathbf{s}^{(k)}) = \sum_{j=1}^{n} \mathrm{EBO}_j(s_0, s_j^{(k)}), \tag{3.20}$$

and the corresponding total cost of spare engines,

$$C(s_0, \mathbf{s}^{(k)}) = cs_0 + \sum_{j=1}^{n} cs_j^{(k)} = c(s_0 + k), \tag{3.21}$$

where the last equality follows from the fact that in each iteration of the marginal allocation algorithm exactly one more spare engine is allocated, and $\mathbf{s}^{(0)} = (0, \cdots, 0)^T$. This means that it is sufficient to calculate $\mathbf{s}^{(k)}$ for $k = 0, 1, \cdots, s^{\max} - s_0$.

The generated efficient solutions are saved, and also the following EBO-values:

$$F(s_0, k) = \mathrm{EBO}(s_0, \mathbf{s}^{(k)}), \text{ for } k = 0, 1, \cdots, s^{\max} - s_0. \tag{3.22}$$

3.5 The complete algorithm for Model 3

Start with $s_0 = 0$, and apply the algorithm described above (for fixed s_0).
This gives a set of efficient solutions for the case $s_0 = 0$, and corresponding EBO-values:

$$F(0,0), F(0,1), \cdots, F(0, s^{\max}). \tag{3.23}$$

The restricted efficient curve for the case $s_0 = 0$ is then the piecewise linear curve between the $s^{\max} + 1$ points

$$(0, F(0,0)), \, (c, F(0,1)), \, \cdots, (s^{\max} c, F(0, s^{\max})). \tag{3.24}$$

Then let $s_0 = 1$, and apply the algorithm described above (for fixed s_0).
This gives a set of efficient solutions for the case $s_0 = 1$, and corresponding EBO-values:

$$F(1,0), F(1,1), \cdots, F(1, s^{\max} - 1). \tag{3.25}$$

The restricted efficient curve for the case $s_0 = 1$ is then the piecewise linear curve between the $s^{\max}$ points

$$(c, F(1,0)), \, (2c, F(1,1)), \, \cdots, (s^{\max} c, F(1, s^{\max} - 1)). \tag{3.26}$$

This is repeated for $s_0 = 2, \cdots, s^{\max}$.
Note that the restricted efficient curve for the case $s_0 = s^{\max}$ consists of a single point $(s^{\max} c, F(s^{\max}, 0))$.

We have now obtained $s^{\max} + 1$ curves, each corresponding to a fixed value on s_0 A natural curve for the complete model, where s_0 is not fixed, is then the piecewise linear curve between the $s^{\max} + 1$ points

$$(0, F(0)), \, (c, F(1)), \, \cdots, (s^{\max} c, F(s^{\max})), \tag{3.27}$$

where

$$F(\ell) = \min_{s_0} \{F(s_0, \ell - s_0) \,|\, 0 \leq s_0 \leq \ell\}, \; \ell = 0, 1, \cdots, s^{\max}. \tag{3.28}$$

Note that $F(\ell)$ is the minimum value of $\text{EBO}_0(s_0, \mathbf{s})$ if $C(s_0, \mathbf{s})$ is required to be $\leq c\ell$.
If this curve is convex, then it is also the efficient curve for Model 3. Otherwise, the efficient curve is obtained by generating the southwestern boundary of the convex hull of the $s^{\max}+1$ points in (3.27). This is an easy task.

3.6 Several LRU in Model 3

It is easy to extend the model above to cover the case where there are several different types of LRU:s, denoted $\text{LRU}_1, \ldots, \text{LRU}_m$.

In that case, one considers one LRU_j at a time, and determines the efficient curve for each LRU_j using the method described above. This results in m curves. Thereafter, we determine a total efficient curve based on all the LRU_j:s. This is done by marginal allocation, *i.e.*, the total efficient curve is constructed from line segments from the previously generated efficient curves of the individual LRU_j:s (It is constructed from left to right, starting with no inventory and then adding the LRU corresponding to the first steepest segment, then the second-steepest, etc.).

4 Model 4 (one base, one LRU, several SRU)

In this *Multi-indenture* model we only have one type of LRU, here referred to as *aircraft-engine*, but I different types of SRU:s (shop-replaceable units), called $\text{SRU}_1, \ldots, \text{SRU}_I$.

When an airplane with a defect engine arrives at a base, the engine is immediately replaced by a new engine from the inventory of spare engines, unless the inventory is empty. If there are no engines in the inventory that can replace the defect engine, a back order is generated at the base. The aircraft from which the defect engine originated then becomes temporarily inoperative.

Aircrafts with an engine (LRU) in need of repair arrive according to a Poisson process with intensity λ_0. Each defect LRU is assumed to have an error on exactly one of its I SRU:s, which means that defect units of SRU_i arrive to the base according to a Poisson process with intensity λ_i , where $\lambda_1 + \ldots + \lambda_I = \lambda_0$.

The process of repairing a defect engine is assumed to consist of two steps. Step 1 consists of troubleshooting in order to identify the defect SRU_i and taking apart the engine to remove that particular SRU_i. Step 2 consists of taking a functioning SRU_i from the inventory, mounting it and conducting subsequent tests to make sure the engine works. If there are no units in the inventory of the requested SRU_i there will be a resulting back order of SRU_i. Defect SRU:s are repaired. The repair times are assumed to be independent stochastic random variables.

Notations:

T_i = The average repair time for SRU_i .
T_0 = The repair time for a defect engine, provided that the SRU_i that is needed is available in the inventory. (We assume that this time is independent of i)
$T_0 = T_0^{(1)} + T_0^{(2)}$, where $T_0^{(i)}$ is the part of the total repair time T_0 that corresponds to Step i in the repair process (see above).

X_i = the number of defect SRU_i:s in the workshop.
X_0 = the number of defect engines in the workshop.
s_i = the number of spare units of SRU_i .
s_0 = the number of spare units of LRU (*i.e.*, engines).

Since the repair times for SRU_i are independent (and equally distributed for a fixed i) it follows from Palm's theorem that X_i is a Poisson distributed random variable with $\text{E}[X_i] = \lambda_i T_i$. This means that $\text{EBO}_i(s_i) = \text{E}[(X_i - s_i)^+]$, *i.e.*, the average number of back orders of SRU_i, can be computed with the same type of recursive equations that was used in Model 1. If needed, one can also compute $\text{VBO}_i(s_i) = \text{Var}[(X_i - s_i)^+]$.

Now consider X_0, which can be written as $X_0 = Y + Z$, where

Y = the number of defect engines that have been received at the base in the last T_0 time units,

Z = the number of defect engines that were received at the base more than T_0 time units ago, but that were still waiting for a SRU $T_0^{(2)}$ time units ago.

Y is Poisson distributed with expected value and variance = $\lambda_0 T_0$.

Z has the same distribution as $\sum_{i=1}^{I}(X_i - s_i)^+$, because every back order of some SRU$_i$ gives rise to a prolonged waiting time for some defect engine.

This leads to

$$\mathrm{E}[Z] = \sum_{i=1}^{I} \mathrm{EBO}_i(s_i)\,. \tag{4.1}$$

Moreover, the number of back orders of various SRU$_i$:s are independent. Hence,

$$\mathrm{Var}[Z] = \sum_{i=1}^{I} \mathrm{VBO}_i(s_i)\,. \tag{4.2}$$

Thus, it follows that

$$\mathrm{E}[X_0] = \lambda_0 T_0 + \sum_{i=1}^{I} \mathrm{EBO}_i(s_i)\,, \tag{4.3}$$

$$\mathrm{Var}[X_0] = \lambda_0 T_0 + \sum_{i=1}^{I} \mathrm{VBO}_i(s_i)\,. \tag{4.4}$$

In the METRIC model one makes the assumption (approximation) that the repair times for defect engines are independent random variables, which after using Palm's theorem gives that X_0 is a Poisson distributed random variable with expected value given by (4.3). This, in turn, gives that it is possible to compute $\mathrm{EBO}_0(s_0) = \mathrm{E}[(X_0 - s_0)^+]$, *i.e.*, the average number of inoperative aircrafts at the base, by using the same type of recursive equations that was used in Model 1. However, note that since $\mathrm{E}[X_0]$ depends on $\mathbf{s} = (s_1, \ldots, s_I)^\mathsf{T}$, so will $\mathrm{EBO}_0(s_0)$. We will therefore use the notation $\mathrm{EBO}_0(s_0, \mathbf{s})$.

If the assumption of the METRIC model had been correct, then the variance and the expected value of X_0 would have been the same. However, this is not the case if at least one $s_i > 0$, since

$$\mathrm{VBO}_i(s_i) > \mathrm{EBO}_i(s_i), \quad \text{for } s_i > 0, \tag{4.5}$$

and hence, according to (4.3) and (4.4),

$$\mathrm{Var}[X_0] > \mathrm{E}[X_0], \quad \text{if any } s_i > 0. \tag{4.6}$$

Still, if $s_i = 0$ for all $i = 1, \ldots, I$, then, according to above, it holds that

$$\mathrm{Var}[X_0] = \mathrm{E}[X_0] = \lambda_0 T_0 + \sum_{i=1}^{I} \lambda_i T_i\,, \quad \text{if all } s_i = 0. \tag{4.7}$$

In the VARI-METRIC model one instead assumes that if any $s_i > 0$ then X_0 is a translated negative binomial distributed random variable with expected value given by (4.3) *and* variance given by (4.4). More specifically, one assumes that if at least one $s_i > 0$ then

$$P(X_0 = 0) = \left(\frac{1}{V_0}\right)^{\frac{\mu_0}{V_0 - 1}}, \quad \text{and} \tag{4.8}$$

$$P(X_0 = k+1) = \frac{\mu_0 + (V_0 - 1)k}{V_0(k+1)} \cdot P(X_0 = k), \quad k = 0, 1, 2, \ldots \tag{4.9}$$

where $\mu_0 = \mathrm{E}[X_0]$ and $V_0 = \dfrac{\mathrm{Var}[X_0]}{\mathrm{E}[X_0]}$.

If all $s_i = 0$, it is still assumed that X_0 is a Poisson distributed random variable with expected value and variance given by (4.7).

It is now possible to compute $\mathrm{EBO}_0(s_0, \mathbf{s}) = \mathrm{E}[(X_0 - s_0)^+]$, *i.e.*, the average number of inoperative aircrafts at the base, by using almost the same type of recursive equations as in the METRIC model. The only difference is that one now uses (4.8) and (4.9) in the computation of the probabilities for the number of engines in the workshop (if some $s_i > 0$).

Efficient curve for Model 4

We will here describe a heuristic method for determining the efficient curve of Model 4. There is a theoretical risk of missing one or a few efficient solutions, but in practice, this rarely happens. We assume that the order that the SRU:s are distributed in does not depend on how many LRU:s we have. So first this order is determined and then in a second step the LRU inventory level is determined.

Let c_0 = the cost per spare engine and let $\mathbf{c} = (c_1, \ldots, c_I)^\mathsf{T}$, where c_i = the cost per spare unit of SRU_i.

First determine the efficient curve for the distribution of spare units of different SRU:s. This is done using the marginal allocation algorithm from Model 2. The result is a piecewise linear convex curve in a coordinate system where the horizontal axis shows $\mathbf{c}^\mathsf{T}\mathbf{s}$ while the vertical axis shows $\sum \mathrm{EBO}_i(s_i)$. Furthermore, the corresponding efficient solutions, here denoted $\mathbf{s}^{(k)}$, $k = 0, 1, 2, \ldots$, where the first one is $\mathbf{s}^{(0)} = (0, \ldots, 0)^T$, are obtained.

One now set $\mathbf{s} = \mathbf{s}^{(0)}$ and computes $\mathrm{EBO}_0(s_0, \mathbf{s}^{(0)})$ for growing values of s_0, starting at $s_0 = 0$. This gives a piecewise linear curve in a coordinate system where the horizontal axis shows $c_0 s_0 + c^T\mathbf{s}^{(0)}$ while the vertical axis shows $\mathrm{EBO}_0(s_0, \mathbf{s}^{(0)})$. The curve is saved.

Next, one set $\mathbf{s} = \mathbf{s}^{(1)}$ and computes $\mathrm{EBO}_0(s_0, \mathbf{s}^{(1)})$ for growing values of s_0, starting at $s_0 = 0$. This gives a new piecewise linear curve, this time in a coordinate system where the horizontal axis shows $c_0 s_0 + \mathbf{c}^T\mathbf{s}^{(1)}$ while the vertical axis shows $\mathrm{EBO}_0(s_0, \mathbf{s}^{(1)})$. This curve is also saved.

The procedure is repeated for $\mathbf{s} = \mathbf{s}^{(2)}$, $\mathbf{s} = \mathbf{s}^{(3)}$, etc., until $\mathbf{c}^T\mathbf{s}^{(k)} > C^{\max}$.

One has now obtained a number of curves, each one corresponding to a given $\mathbf{s} = \mathbf{s}^{(k)}$. Finally, the *lower convex envelope*(also referred to as the *convex hull*) corresponding to these curves is constructed. The obtained piecewise linear convex curve is the efficient curve of Model 4.

5 Model 5 (a central depot and several bases, one LRU, several SRU)

In Model 5, we will combine Models 3 and 4.

As in Model 4, we here have one type of LRU, referred to as "aircraft engine", with I different SRU:s, referred to as $\text{SRU}_1, \ldots, \text{SRU}_I$. As in Model 3, we have J bases, each with a local spare part supply (for engines), and one central depot with a repair workshop and a central spare part supply (for engines and SRU:s).

As no repairs are done at the bases, Model 4 will only enter in the modeling of the depot.

Decision variables in the model:

s_{0j} = the number of spare engines at base j.
s_{00} = the number of spare engines at the depot.
s_{i0} = the number of spare units of SRU_i at the depot.

Parameters of the model:

λ_{0j} = the intensity with which defect engines arrive at base j.
λ_{00} = the intensity with which defect engines arrive to the workshop.
λ_{i0} = the intensity with which defect units of SRU_i arrive to the workshop.

(λ_{0j} and λ_{00} correspond to the parameters λ_j and λ_0 in Model 3, while λ_{i0} corresponds to the parameter λ_i in Model 4. It holds that $\sum_j \lambda_{0j} = \lambda_{00}$ and $\sum_i \lambda_{i0} = \lambda_{00}$.)

T_{i0} = the average repair time for SRU_i.
T_{bd} = the time it takes to transport a defect engine from a base to the depot.
T_{db} = the time it takes to transport a functioning engine from the depot to a base.
T_{rep} = the repair time for a defect engine, provided that the SRU_i that is needed is available in the inventory.
$T_{00} = T_{bd} + T_{rep}$.

Stochastic variables in the model:

X_{i0} = the number of defect SRU_i in the workshop.
X_{00} = the number of defect engines in, or on their way to the workshop.
X_{0j} = the number of defect engines that have been sent from base j to the workshop, but for which replacement engines have still not been received at the local inventory at base j (= the number of engines in the pipeline at base j).

The expected values for these random variables can be computed in exactly the same way as the corresponding expected values in Models 3 and 4. The following result is obtained.

$$\mathrm{E}[X_{i0}] = \lambda_{i0} T_{i0} \,, \tag{5.1}$$

$$\mathrm{E}[X_{00}] = \lambda_{00} T_{00} + \sum_{i=1}^{I} \mathrm{EBO}_{i0}(s_{i0}) \,, \tag{5.2}$$

$$\mathrm{E}[X_{0j}] = \lambda_{0j} (T_{db} + \frac{\mathrm{EBO}_{00}(s_{00})}{\lambda_{00}}) \,. \tag{5.3}$$

Efficient curve for Model 5

Introduce the following vectors to describe the distribution of spares:

$s_{*0} = (s_{10}, \ldots, s_{I0})$, $s_{0*} = (s_{01}, \ldots, s_{0J})$ and $\mathbf{s} = (s_{*0} \,, s_{00} \,, s_{0*})$.

Furthermore, introduce the following functions of $\mathbf{s}$:

$F(\mathbf{s}) = \sum_j \mathrm{EBO}_{0j}$ and $C(\mathbf{s}) = \sum_i c_{i0} s_{i0} + c_{00} s_{00} + \sum_j c_{0j} s_{0j}$.

These will be the two objective functions of our multiobjective optimization problem.

We now wish to determine the efficient curve (and the associated efficient solutions) for the given model, *i.e.*, for the convex curve, with $F(\mathbf{s})$ on the vertical axis and $C(\mathbf{s})$ on the horizontal axis, that describes the minimal average number of inoperative aircrafts as a function of resources invested in spare units.

It is generally very difficult to implement this if one wants to *prove* that the generated curve is correct. However, the following method will in most cases give a very good approximation of the correct curve.

Next we present a three step procedure based on determining the distribution of the SRU:s first, then LRU:s at the depot and SRU:s and finally determining also the distribution of LRU:s at the bases.

Step 1:

First assume that $\mathbf{s} = (s_{*0} \,, 0 \,, 0)$, *i.e.*, $s_{00} = 0$ and $s_{0j} = 0$ for all j.

Then $\mathrm{EBO}_{00} = \mathrm{E}[X_{00}]$ and $\mathrm{EBO}_{0j} = \mathrm{E}[X_{0j}]$.

Given the equations for the expected values we get:

$$F(\mathbf{s}) = \lambda_{00}(T_{bd} + T_{rep} + T_{db}) + \sum_{i=1}^{I} \mathrm{EBO}_{i0}(s_{i0}) \,. \tag{5.4}$$

The assumption of independent workshop times gives that X_{i0} is a Poisson distributed random variable with $\mathrm{E}[X_{i0}] = \lambda_{i0} T_{i0}$. Hence, $\mathrm{EBO}_{i0}(s_{i0})$ can easily be computed.

The first step in the method is to apply margin allocation to spare units of SRU:s at the depot, *i.e.*, to gradually increase the values of the variables s_{i0} in the order that gives the largest reduction of $\sum_i \mathrm{EBO}_{i0}(s_{i0})$ per invested monetary unit. This results in a convex curve, with $F(s_{*0}\,,\,0\,,\,0)$ on the vertical axis and $C(s_{*0}\,,\,0\,,\,0)$ on the horizontal axis.

At the same time, a set of efficient solutions $\{\mathbf{s}_{*0}^{(k)}\}$, $k = 1, 2, 3, \ldots$ is obtained.

Step 2:

Now assume that $\mathbf{s} = (\mathbf{s}_{*0}^{(k)}\,,\,s_{00}\,,\,0)$, *i.e.*, $s_{*0} = \mathbf{s}_{*0}^{(k)}$ = is one of the efficient solutions from Step 1, and $s_{0j} = 0$ for all j.

Then $\mathrm{EBO}_{0j} = \mathrm{E}[X_{0j}]$, and with the given equations for expected values inserted we get

$$F(\mathbf{s}) = \lambda_{00} T_{db} + \mathrm{EBO}_{00}^{(k)}(s_{00})\,, \tag{5.5}$$

where $\mathrm{EBO}_{00}^{(k)}(s_{00})$ is $\mathrm{EBO}_{00}(s_{00})$ for the case when $s_{*0} = \mathbf{s}_{*0}^{(k)}$.

The assumption on independent repair times gives that X_{00} is a Poisson distributed random variable with expected value given by (5.2). This, in turn, makes it possible to compute $\mathrm{EBO}_{00}^{(k)}(s_{00})$ quite easily.

Now, for each point $\mathbf{s}_{*0}^{(k)}$, $\mathrm{EBO}_{00}^{(k)}(s_{00})$ is computed for increasing values of s_{00}, starting from $s_{00} = 0$.

For each point $\mathbf{s}_{*0}^{(k)}$, the procedure renders a convex curve with $F(\mathbf{s}_{*0}^{(k)}\,,\,s_{00}\,,\,0)$ on the vertical axis and $C(\mathbf{s}_{*0}^{(k)}\,,\,s_{00}\,,\,0)$ on the horizontal axis.

The convex hull of the complete set of curves is then computed.

This results in a new convex curve, with $F(s_{*0}\,,\,s_{00}\,,\,0)$ on the vertical axis and $C(s_{*0}\,,\,s_{00}\,,\,0)$ on the horizontal axis.
At the same time, a set of efficient solutions $\{(s_{*0}\,,\,s_{00})^{(\ell)}\}$, $\ell = 1, 2, 3, \ldots$ are obtained.

Note that so far, the procedure is exactly the same as for Model 4.

Step 3:

Now assume that $\mathbf{s} = ((s_{*0}\,,\,s_{00})^{(\ell)}\,,\,s_{0*})$, i.e., $(s_{*0}\,,\,s_{00}) = (s_{*0}\,,\,s_{00})^{(\ell)}$ = one of the efficient solutions from Step 2.

Then it holds that

$$F(\mathbf{s}) = \sum_j \mathrm{EBO}_{0j}^{(\ell)}(s_{0j})\,, \tag{5.6}$$

where $\mathrm{EBO}_{0j}^{(\ell)}(s_{0j})$ is $\mathrm{EBO}_{0j}(s_{0j})$ for the case $(s_{*0}\,,\,s_{00}) = (s_{*0}\,,\,s_{00})^{(\ell)}$.

The assumption on independent pipeline times gives that each X_{0J} is a Poisson distributed random variable with expected value given by (5.3). From this fact, it follows that $\mathrm{EBO}_{0j}^{(\ell)}(s_{0j})$, can be easily computed.

For each point $(s_{*0}\,,\, s_{00})^{(\ell)}$ one now carry out a marginal allocation of spare engines at the bases, i.e., one gradually increases the values of the variables s_{0j} in the order which gives the greatest reduction in $\sum_j \mathrm{EBO}_{0j}^{(\ell)}(s_{0j})$ per invested money unit.

For each point $(s_{*0}\,,\, s_{00})^{(\ell)}$ one obtains a convex curve, with $F((s_{*0}\,,\, s_{00})^{(\ell)}\,,\, s_{0*})$ on the vertical axis and $C((s_{*0}\,,\, s_{00})^{(\ell)}\,,\, s_{0*})$ on the horizontal axis.

The convex hull for the complete set of curves is computed.

This gives a new convex curve, with $F(s_{*0}\,,\, s_{00}\,,\, s_{0*})$ on the vertical axis and $C(s_{*0}\,,\, s_{00}\,,\, s_{0*})$ on the horizontal axis. At the same time, a set of efficient solutions $\{\mathbf{s}^{(q)}\} = \{(s_{*0}\,,\, s_{00}\,,\, s_{0*})^{(q)}\}$, for $q = 1, 2, 3, \ldots$, is obtained.

This is our requested efficient curve and our efficient solutions.

5.1 Generalization of Model 5 to several LRU

We now extend Model 5, by assuming that there are several different types of LRU:s, each with an associated set of SRU:s, and that an airplane gets grounded if there is a back order on any of these LRU:s at some base.

The efficient curve for this extended model is as before the convex curve that describes the minimal average number of inoperative aircrafts as a function of resources invested in spare units. However, we will now have $\sum_k \sum_j \mathrm{EBO}_{0jk}$ on the vertical axis and $\sum_k C_k(s_k)$ on the horizontal axis, where EBO_{0jk} denotes the average number of back orders of LRU:s of type k at base j, while $C_k(s_k)$ denotes the cost of spare units of LRU:s of type k, and the belonging SRU:s.

This efficient curve can be obtained as follows:

For each LRU family (*i.e.*, an LRU-type with its associated SRU:s) the Steps 1-3 in Model 5 above are performed. The result is a convex curve for each LRU family.

Then, margin allocation based on these curves is done. This means that line segments from the obtained curves (first the steepest segment, then the second steepest, etc..) are used to construct a new convex curve.

This new convex curve is the efficient curve of the extended model. At the same time, one has also received a set of efficient solutions.

6 APPENDIX: Palm's Theorem

Theorem: Assume that defect units arrive to a workshop according to a Poisson process with an intensity of λ units per time unit. Furthermore, assume that the repair times for the defect units are independent, equally distributed stochastic random variables with expected value T time units. Then, the number of defect units in the workshop is a Poisson distributed random variable with expected value λT units.

Remark: The " Repair Time" is the time *from* that a malfuctioning item arrives at the workshop *until* the same item has been repaired and leaves the workshop.

Sketch of Proof:

Let τ denote the repair time for a defect unit. We will only prove the Theorem for the case where τ is a discrete stochastic random variable with finite range space $\{t_1, \ldots, t_N\}$ (as this range space can be chosen arbitrarily "close", it is not hard to believe that the theorem would apply in the general case as well). Assume that $t_1, \ldots, t_N$ are known and represent possible repair times and that $p_i = P(\tau = t_i)$ are the corresponding probabilities which are also known and satisfy $\sum p_i = 1$ and $\sum p_i t_i = T$.

We can then consider the situation as follows: Defect units arrive according to a Poisson process with intensity λ. For each arriving unit, the length of the repair time is picked randomly. If the result of the randomization is that the repair time will be t_i (which occurs with probability p_i) then the unit is placed in the "i:th sub-workshop" which has a deterministic repair time $= t_i$. When the unit leaves this sub-workshop t_i time units later, it leaves the actual workshop as well.

It is a well known property of Poisson processes that the procedure above leads to defect units arriving at the N different sub-workshops according to independent Poisson processes with intensities $\lambda_i = p_i m$, for $i = 1, \ldots, N$.

Let $X_i =$ the number of units in the i:th sub-workshop. Then $X_i =$ the number of units that arrived to the i:th sub-workshop during the last t_i time units. According to another well-known property of Poisson processes, this number is a Poisson distributed random variable with expected value $\lambda_i t_i$.

Let X = the number of units in the real workshop. As each arriving unit is placed in of one sub-workshop, it follows that $X = \sum X_i$, where the X_i:s according to above are independent Poisson distributed random variables. According to a well-known property of Poisson distributions, this implies that X is a Poisson distributed random variable with expected value $\sum \lambda_i t_i = m \sum p_i t_i = mT$, which is exactly what we wanted to show.