Abstract

When using the Pontryagin Maximum Principle in optimal control problems, the most difficult part of the numerical solution is associated with the non-linear operation of the maximization of the Hamiltonian over the control variables. For a class of problems, the optimal control vector is a vector function with continuous time derivatives. A method is presented to find this smooth control without the maximization of the Hamiltonian. Three illustrative examples are considered.
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The classical optimal control problem

- Consider the classical optimal control problem (OCP), Pontryagin et al. (1962), Lee and Marcus (1967), Athans and Falb (1966), etc.

\[
\int_0^T f_0(x,u)dt \rightarrow \min
\]  

(1)

\[
\frac{dx}{dt} = f(x,u),
\]  

(2)

\[
x(0) = x_0, \ x(T) = x_T.
\]  

(3)

- the control variables \(u(t) \in \mathbb{R}^m\), the state variables \(x(t) \in \mathbb{R}^n\), and \(f(x,u) \in \mathbb{R}^n\) are column vectors, with \(m \leq n\).

- \(f_0(x,u), f(x,u)\) are smooth in all arguments.

- The Hamiltonian is

\[
H = p^T f(x,u) - f_0(x,u).  
\]  

(5)

where it holds for the column vector \(p(t) \in \mathbb{R}^n\) of co-state variables, that

\[
\frac{dp}{dt} = -\frac{\partial H}{\partial x} = -\frac{\partial f^T}{\partial x} p + \frac{\partial f_0^T}{\partial x}
\]  

(6)

according to the Pontryagin Maximum Principle (PMP).
Classical solution

- If an optimal solution \((x^*, u^*, p^*)\) exists, then, by PMP, it holds that \(H(x^*, u^*, p^*) \geq H(x^*, u, p^*)\) implying here by smoothness, and the presence of constraint (3) only, that for \(u = u^*\),

\[
\frac{\partial H}{\partial u} = 0. \tag{7}
\]

or, with (5) inserted into (7),

\[
p^T \frac{\partial f}{\partial u} - \frac{\partial f_0}{\partial u} = 0 \tag{8}
\]

where \(\frac{\partial f_0}{\partial u}\) is 1\(\times\)m, and \(\frac{\partial f}{\partial u}\) is n\(\times\)m.

- To find \((x^*, u^*, p^*)\) the two point boundary value problem (2)-(6) must be solved.

- At each \(t\), (8) gives \(u^*\) as a function of \(x\) and \(p\). (8) is often non-linear, and computationally costly.

- \(p(0)\) has as many unknowns as given end conditions \(x(T)\).
The new idea without optimization w.r.t. $u$

- We note that (8) is linear in $p$.
- Assume that $\text{rank}(\partial f/\partial u)=m \Rightarrow \exists$ a non-singular $m \times m$ submatrix. Then, re-index the corresponding vectors

$$x = [x^a; x^b]; p = [p^a; p^b]; f(x, u) = [f^a(x, u); f^b(x, u)].$$

where $\Box^a$ denotes an $m$-vector. Then, (8) gives

$$p^a_T \frac{\partial f^a}{\partial u} + p^b_T \frac{\partial f^b}{\partial u} - \frac{\partial f_0}{\partial u} = 0 \tag{9}$$

$$p^a = -\left(\frac{\partial f^a}{\partial u}\right)^{-1} \frac{\partial f^b}{\partial u} p^b + \left(\frac{\partial f^a}{\partial u}\right)^{-1} \frac{\partial f_0}{\partial u} = A(x, p^b, u) \tag{10}$$

- Hence by linear operations, $m$ elements of $p \in \mathbb{R}^n$, i.e. $p^a$, are computed as a function of $u$, $x$, and $p^b$. 

\[ \int_0^T f_0(x, u)dt \rightarrow \min \tag{1} \]

\[ \frac{dx}{dt} = f(x, u), \tag{2} \]

\[ x(0) = x_0, \ x(T) = x_T. \tag{3} \]

\[ H = p^T f(x, u) - f_0(x, u). \tag{5} \]

\[ \frac{dp}{dt} = -\frac{\partial H}{\partial x} = -\frac{\partial f^T}{\partial x} p + \frac{\partial f_0^T}{\partial x} \tag{6} \]

\[ \frac{\partial H}{\partial u} = 0. \tag{7} \]

\[ p^T \frac{\partial f}{\partial u} - \frac{\partial f_0}{\partial u} = 0 \tag{8} \]
The new idea, cont’d

\[ \int_0^T f_0(x, u) dt \rightarrow \min \]  

(1)

\[ \frac{dx}{dt} = f(x, u), \]  

(2)

\[ x(0) = x_0, \quad x(T) = x_T. \]  

(3)

\[ H = p^T f(x, u) - f_0(x, u). \]  

(5)

\[ \frac{dp}{dt} = -\frac{\partial H}{\partial x} + \frac{\partial f}{\partial x} p + \frac{\partial f_0}{\partial x} \]  

(6)

\[ \frac{\partial H}{\partial u} = 0. \]  

(7)

\[ p^a \frac{\partial f^a}{\partial u} + p^b \frac{\partial f^b}{\partial u} - \frac{\partial f_0}{\partial u} = 0 \]  

(9)

\[ p^a \left[ \frac{\partial f^a}{\partial u} \right] - p^b \left[ \frac{\partial f^b}{\partial u} \right] \triangle A(x, p^b, u) \]  

(10)

- Differentiate (10):

\[ \frac{dp^a}{dt} = \frac{\partial A}{\partial x} f(x, u) + \frac{\partial A}{\partial u} \frac{du}{dt} + \frac{\partial A}{\partial p^b} \frac{dp^b}{dt} \]  

(11)

\[ B = \frac{\partial A}{\partial u} \]

- where \( B \) is assumed non-singular. (6) gives

\[ \frac{dp^a}{dt} = -\frac{\partial H}{\partial x^a} = -\frac{\partial f^a}{\partial x^a} p^a - \frac{\partial f^b}{\partial x^a} p^b + \frac{\partial f_0}{\partial x^a} \]  

(12)

\[ \frac{dp^b}{dt} = -\frac{\partial H}{\partial x^b} = -\frac{\partial f^a}{\partial x^b} p^a - \frac{\partial f^b}{\partial x^b} p^b + \frac{\partial f_0}{\partial x^b} = S(x, p^b, u) \]  

(13)

- (10) into RHS of (12, 13), noting that \( dp^a/dt \) is given by the RHS of (11) and (12), and solving for \( du/dt \), gives

\[ \frac{du}{dt} = B^{-1} \left[ -\frac{\partial f^a}{\partial x^a} A - \frac{\partial f^b}{\partial x^a} \right] p^b + \frac{\partial f_0}{\partial x^a} - \frac{\partial A}{\partial x} f(x, u) - \frac{\partial A}{\partial p^b} \frac{dp^b}{dt} \right] \]  

(14)
**Theorem**: If the optimal control problem (1)-(3), m ≤ n, has the optimal solution \( x^*, u^* \) such that \( u^* \) is smooth and belongs to the open set \( U \), and if the Hamiltonian is given by (5), the Jacobians \( \frac{\partial f^a}{\partial u} \) and \( B = \frac{\partial p^a}{\partial u} \) are non-singular, then the optimal states \( x^* \), co-states \( p^{*b} \), and control \( u^* \) satisfy

\[
\begin{align*}
\frac{du}{dt} &= F(x, p^b, u), \\
\frac{dx}{dt} &= f(x, u), \\
\frac{dp^b}{dt} &= S(x, p^b, u). \\
\end{align*}
\] (15)

with the appropriate initial conditions \( u(0) = u_0, p^b(0) = p^b_0 \) to be found.

**Remark**: if \( m = n \), then \( x^a = x \), and \( p^a = p \), and (15) becomes

\[
\begin{align*}
\frac{dx}{dt} &= f(x, u) \\
\frac{du}{dt} &= F(x, u). \\
\end{align*}
\] (15′)

**Remark**: The number of equations in (15) is 2n, just as in PMP, but without the maximization of the Hamiltonian.
Example 1: Rigid body rotation

Stopping axisymmetric rigid body rotation (Athans and Falb, 1963)

\[
\begin{align*}
\frac{dx}{dt} & = ay + u_1, \\
\frac{dy}{dt} & = -ax + u_2
\end{align*}
\]  
(16)

\[x(T) = 0, y(T) = 0 \]  
(17)

\[J = \frac{1}{4} \int_0^T (u_1^2 + u_2^2)^2 dt \rightarrow \text{min} \]

\[p = A = \begin{bmatrix} u_1(u_1^2 + u_2^2) \\ u_2(u_1^2 + u_2^2) \end{bmatrix} \]

\[\frac{dp_x}{dt} = ap_y \]

\[\frac{dp_y}{dt} = -ap_x \]

\[B = \begin{pmatrix} 3u_1^2 + u_2^2 & 2u_1u_2 \\ 2u_1u_2 & u_1^2 + 3u_2^2 \end{pmatrix} \]

\[B^{-1} = \frac{1}{(3[u_1^2 + u_2^2]_2)} \begin{pmatrix} u_1^2 + 3u_2^2 & -2u_1u_2 \\ -2u_1u_2 & 3u_1^2 + u_2^2 \end{pmatrix} \]

\[\frac{du_1}{dt} = au_2, \]

\[\frac{du_2}{dt} = -au_1 \]

(23)
Example 1: Rigid body rotation, cont’d

The problem is solved without maximizing the Hamiltonian!

\[ \begin{align*}
\frac{dx}{dt} & = ay + u_1, \\
\frac{dy}{dt} & = -ax + u_2 \\
x(T) &= 0, y(T) = 0
\end{align*} \]  
(16)

\[ \begin{align*}
\frac{du_1}{dt} & = au_2, \\
\frac{du_2}{dt} & = -au_1
\end{align*} \]  
(23)

- Polar coordinates:
  \[ \begin{align*}
x & = r \sin \theta \\
y & = r \cos \theta
\end{align*} \]  
(24)

- then, clearly,
  \[ \begin{align*}
u_1 & = -C \frac{x}{\sqrt{x^2 + y^2}} \\
u_2 & = -C \frac{y}{\sqrt{x^2 + y^2}}
\end{align*} \]  
(25)

\[ \frac{dr}{dt} = -C' \]
\[ \frac{d\theta}{dt} = a. \]  
(26)

\( \Rightarrow (u_1, u_2) \) and \((x, y)\) rotate collinearly with the same angular velocity \(a\).

- Let
  \[ \sqrt{x(0)^2 + y(0)^2} = R \]

then, from (17), (25), (26),

\[ \begin{align*}
u_1(0) / C' & = -x(0) / R \\
u_2(0) / C' & = -y(0) / R
\end{align*} \]

\[ C = R / T \]

The problem is solved without maximizing the Hamiltonian!
Example 2: Optimal spacing for greenhouse lettuce growth

Optimal variable spacing policy (Seginer, Ioslovich, Gutman), assuming constant climate

$$\frac{dv}{dt} = \frac{v}{W} G(W),$$

$$J = \int_0^T \frac{v}{W} c_R dt$$

$$v(T) = v_T$$

$$v = a W$$

with \(v\) [kg/plant] = dry mass, \(G\) [kg/m²/s] net photosynthesis, \(W\) [kg/m²] plant density (control), \(a\) [m²/plant] spacing, \(v_T\) marketable plant mass, and final time \(T\) [s] free.

$$H = \frac{v}{W} (pG(W) - c_R)$$  \hspace{1cm} (32)

$$\frac{\partial H}{\partial W} = -\frac{v}{W^2} (pG(W) - c_R) + \frac{vp}{W} \frac{\partial G}{\partial W}$$

- \(p\) is obtained from
  $$\frac{\partial H}{\partial W} = 0,$$

$$p = \frac{c_R}{G(W) - W \frac{\partial G}{\partial W}}$$  \hspace{1cm} (34)

- \(\frac{dp}{dt} = -\frac{\partial H}{\partial v} = -\frac{pG(W) - c_R}{W}$$  \hspace{1cm} (35)

- \(\frac{dp}{dt} = -\frac{c_R \frac{\partial G}{\partial W}}{G - W \frac{\partial G}{\partial W}}$$  \hspace{1cm} (36)

- Differentiating (34),

$$\frac{dp}{dt} = \frac{dW}{dt} \frac{c_R W \frac{\partial G}{\partial W^2}}{(G - W \frac{\partial G}{\partial W})^2}$$  \hspace{1cm} (37)

- For free final time, (38)... also w/o maximization of the Hamiltonian, I&G 99
Example 3: Maximal area under a curve of given length


\[ H = u + \frac{p_1}{\sqrt{1 + u^2}} + \frac{p_2}{2} \]

(46)

Differentiating (49), and using (47) gives

\[ p_1 = -\frac{p_2}{\sqrt{1 + u^2}} \]

(49)

Guessing \( p_2 = \text{constant} \) and integrate (43), (44), (50), such that (45) is satisfied, yields

\[ u(0) = \frac{1}{\sqrt{3}} \]

(45)

\[ p_2 = -1 \]

(44)

\[ x_1(0) = x_1(1) = 0 \]

(43)

\[ x_2(0) = 1, x(1) = \frac{\pi}{3} \]

(42)

\[ \int_0^1 x_1(\frac{dx_1}{dt}) \rightarrow \max \]

(41)
Conclusions

- A method to find the smooth optimal control for a class of optimal control problems was presented.
- The method does not require the maximization of the Hamiltonian over the control.
- Instead, the ODEs for $m$ co-states are substituted for ODEs for the $m$ smooth control variables.
- Three illustrative examples were given.