KTH Matematik

Homework 1
Mathematical Systems Theory, 5B1742
Spring 2006

1. In this problem we investigate sampling of a harmonic oscillator. The continuous time dynamics is given as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{1}$$

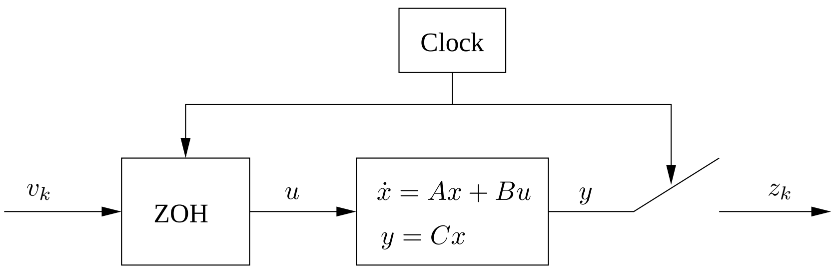
(a) Is the the harmonic oscillator in (1) completely reachable?

Now consider zero-order hold sampling of the harmonic oscillator. Zero order hold sampling is the most common way to do the digital to analog (D/A) and analog to digital (A/D) conversion when implementing a control algorithm in a computer. The output from the linear system is sampled at time instances $t = kh$ to generate the discrete output sequence $\{z_k\}_{k=0}^{\infty}$, where $z_k = x(kh)$ (A/D conversion). The computer algorithm uses these samples to generate a control sequence $\{v_k\}_{k=0}^{\infty}$. This sequence is then fed through the zero order hold circuit, which generates the piece-wise constant control signal (D/A conversion)

$$u(t) = v_k, \qquad t \in [kh, (k+1)h)$$

Zero-order hold sampling is illustrated in Figure 1. The purpose of the clock circuit is to synchronize the zero-order hold circuit and the sampler.

For the control design it is important to have a discrete time equivalent of the sampled system, i.e. we need to determine the dynamics from v to z. This is discussed in Lindquist and Sand on page 14.



Figur 1: Zero-order hold sampling of a harmonic oscillator.

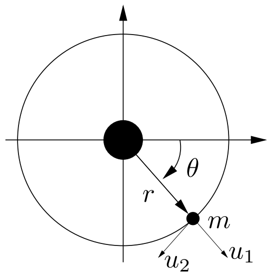
(b) The sampled harmonic oscillator is a discrete time system on the form

$$z_{k+1} = Fz_k + Gu_k \tag{2}$$

Compute F and G.

(c) For what values of the sampling time h is the discrete time system in (2) completely reachable.

2. Consider the classical satellite problem, see Figure 2.



Figur 2: Control of satellite in orbit around the earth.

The satellite is modeled as a particle of unit mass moving in an inverse square law force field. The position of the satellite is given by the radius $r(t)$ and the angle $\theta(t)$. The satellite has the capability of thrusting in the radial direction with a thrust u_1 and thrusting in the tangential direction with a thrust u_2. The equations of motion are

$$\ddot{r} = r\dot{\theta}^2 - \frac{k}{r^2} + u_1$$
$$\ddot{\theta} = -2\frac{1}{r}\dot{r}\dot{\theta} + \frac{1}{r}u_2.$$

(a) Verify that with no input applied, ($u_1 = u_2 = 0$) circular orbits of the form $r(t) = \sigma, \theta(t) = \omega t$, where σ and ω are constants, are possible. What is the relation between k, σ and ω?

(b) Normalize such that $\sigma = 1$ and and show that the linearization of the system along the circular orbit becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

(c) Is the linearized system completely reachable?

(d) Is the system completely reachable if the radial thrust is broken? Is the system completely reachable if the tangential thrust is broken?

(e) Is the system completely observable if only r can be measured? If only θ can be measured?

3. In this problem you will first investigate the controlled system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0 \qquad (\mathcal{C})$$

and the dual observed system

$$\begin{aligned} \dot{z}(t) &= -A(t)^T z(t), \quad z(t_1) = z_1 \\ y(t) &= B(t)^T z(t) \end{aligned} \qquad (\mathcal{O})$$

(a) Prove that the reachability Gramian (corresponding to system $\mathcal{C}$) satisfies the differential equation

$$\frac{d}{dt} W_c(t_0, t) = A(t) W_c(t_0, t) + W_c(t_0, t) A(t)^T + B(t) B(t)^T$$

(b) Let $\Phi_c(t, s)$ be the transition matrix corresponding to the homogeneuous part of the system in $(\mathcal{C})$ (i.e. $\dot{x} = A(t)x$) and $\Phi_o(t, s)$ be the transition matrix corresponding to the homogeneous part of the system in $(\mathcal{O})$. Show that $\Phi_o(t, s) = \Phi_c(s, t)^T$.

(c) Prove that $W_c(t_0, t_1) = \Phi_c(t_1, t_0) M_0(t_0, t_1) \Phi_c(t_1, t_0)^T$, where W_c is the reachability Gramian for $(\mathcal{C})$ and M_o is the observability Gramian for $(\mathcal{O})$.