KTH Matematik

HW1 in Mathematical Systems Theory, 2006
Answers and solution sketches

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1. (a) $\mathcal{R} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, which has full rank. The system is therefore reachable.

(b)

$$F = e^{Ah} = \begin{bmatrix} \cos(h) & \sin(h) \\ -\sin(h) & \cos(h) \end{bmatrix}$$
$$G = \int_0^h e^{At} B dt = \begin{bmatrix} 1 - \cos(h) \\ \sin(h) \end{bmatrix}$$

(c)

$$\mathcal{R} = \begin{bmatrix} G & FG \end{bmatrix} = \begin{bmatrix} 1 - \cos(h) & \cos(h) + \sin(h)^2 - \cos(h)^2 \\ \sin(h) & -\sin(h) + 2\sin(h)\cos(h) \end{bmatrix}$$

We have

$$\det(\mathcal{R}) = \sin(h)(-1 + \cos(h))$$

which is zero if and only if $h = k\pi$, $k = 0, \pm 1, \pm 2 \ldots$. Hence the system is completely reachable when $h \neq k\pi$, $k = 0, \pm 1, \pm 2 \ldots$.

2. (a) If we plug the suggested solution to the dynamic equations we get

$$0 = \sigma\omega^2 - \frac{k}{\sigma^2}$$
$$0 = 0$$

i.e. the system admits a circular solution when $\sigma^3\omega^2 = k$.

(b) Let $\sigma = 1$ and define the states (deviations from the circular soltion)

$$x_1 = r - 1, \quad x_2 = \dot{r}, \quad x_3 = \theta - \omega t, \quad x_4 = \dot{\theta} - \omega$$

Linearization gives the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

(c) The controllability matrix is

$$\mathcal{R} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 2\omega & -\omega^2 & 0 \\ 1 & 0 & 0 & 2\omega & -\omega^2 & 0 & 0 & -2\omega^3 \\ 0 & 0 & 0 & 1 & -2\omega & 0 & 0 & -4\omega^2 \\ 0 & 1 & -2\omega & 0 & 0 & -4\omega^2 & -2\omega^3 & 0 \end{bmatrix}$$

The first four rows are linearly independent and therefore $\mathrm{Im}\mathcal{R} = \mathbf{R^4}$. Hence, the system is completely reachable. To see that the first four columns are linearly independent consider the equation system

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -2\omega & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = 0$$

This implies that $\alpha_3 = \alpha_4 = 0$ (first and third equation) and then the second and fourth row implies $\alpha_1 = \alpha_2 = 0$. This proves the claimed linear independence.

(d) If $u_1 = 0$ (radial thrust broken) we get

$$\mathcal{R} = \begin{bmatrix} B_2 & AB_2 & A^2B_2 & A^3B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2\omega & 0 \\ 0 & 2\omega & 0 & -2\omega^3 \\ 0 & 1 & 0 & -4\omega^2 \\ 1 & 0 & -4\omega^2 & 0 \end{bmatrix}$$

Consider again

$$\begin{bmatrix} 0 & 0 & 2\omega & 0 \\ 0 & 2\omega & 0 & -2\omega^3 \\ 0 & 1 & 0 & -4\omega^2 \\ 1 & 0 & -4\omega^2 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_3 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = 0$$

The first equation shows that $\alpha_3 = 0$. Then equation four gives $\alpha_1 = 0$. The remaining equation system is

$$\begin{bmatrix} 2\omega & -2\omega^3 \\ 1 & -4\omega^2 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \alpha_4 \end{bmatrix} = 0$$

The matrix is invertible, which implies $\alpha_2 = \alpha_4 = 0$. Hence the column vectors of $\mathcal{R}$ are linearly independent and $\mathrm{Im}\,\mathcal{R} = \mathbf{R^4}$ and the system is controllable.

(e) If $u_2 = 0$ then

$$\mathcal{R} = \begin{bmatrix} B_1 & AB_1 & A^2B_1 & A^3B_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -\omega^2 \\ 1 & 0 & -\omega^2 & 0 \\ 0 & 0 & -2\omega & 0 \\ 0 & -2\omega & 0 & 2\omega^3 \end{bmatrix}$$

We see that the first three columns are linearly independent but the fourth column is $-\omega^2$ times the second column. Hence $\dim(\mathrm{Im}\,\mathcal{R}) = 3$ and the system is not controllable.

(d) If only r is measure then

$$\mathcal{O} = \begin{bmatrix} C_1 \\ C_1A \\ C_1A^2 \\ C_1A^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & -\omega^2 & 0 & 0 \end{bmatrix}$$

Clearly $\operatorname{Ker}\mathcal{O} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ so the system is not observable from r. If only θ is measured then

$$\mathcal{O} = \begin{bmatrix} C_2 \\ C_2A \\ C_2A^2 \\ C_2A^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \\ -6\omega^3 & 0 & 0 & -4\omega^2 \end{bmatrix}$$

Now $\operatorname{Ker}\mathcal{O} = \{0\}$ so the system is observable from the angle measurement.

3. We use the definitions

$$\frac{d}{dt}\Phi_c(t,s) = A(t)\Phi_c(t,s), \qquad \Phi(s,s) = I$$
$$\frac{d}{dt}\Phi_o(t,s) = -A(t)^T\Phi_o(t,s), \qquad \Phi_o(s,s) = I$$

(a) The claim follows since differentiation gives

$$\begin{aligned} \frac{d}{dt}W_c(t_0,t) &= \frac{d}{dt}\int_{t_0}^{t} \Phi_c(t,\tau)B(\tau)B(\tau)^T\Phi_c(t,\tau)^T d\tau \\ &= A(t)W_c(t,t_0) + W_c(t,t_0)A(t)^T + B(t)B(t)^T \end{aligned}$$

(b) Using one of the rules derived for the transition matrix we get

$$\frac{d}{dt}\Phi_c(s,t)^T = (\frac{d}{dt}\Phi_c(s,t))^T = (-\Phi_c(s,t)A(t))^T = -A(t)^T\Phi_c(s,t)$$

and therefore $\Phi_o(t,s) = \Phi_c(s,t)^T$.

(c) We have

$$W_c(t_0,t_1) = \int_{t_0}^{t_1} \Phi_c(t_1,\tau)B(\tau)B(\tau)^T\Phi_c(t_1,\tau)^T d\tau$$

For the observability Gramian we have

$$\begin{aligned} M_o(t_0,t_1) &= \int_{t_0}^{t_1} \Phi_o(\tau,t_0)^T C(\tau)^T C(\tau)\Phi_o(\tau,t_0) d\tau \\ &= \int_{t_0}^{t_1} \Phi_c(t_0,\tau)B(\tau)B(\tau)^T\Phi_c(t_0,\tau)^T d\tau \end{aligned}$$

Hence

$$\Phi(t_1,t_0)M_o(t_0,t_1)(\Phi(t_1,t_0)^T = W_c(t_0,t_1)$$