KTH Matematik

HW2 in Mathematical Systems Theory, 2006
Answers and solution sketches

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1. (*a*) The realization is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-\frac{R}{L} & 0\\ 0 & -\frac{1}{RC}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{1}{L}\\ \frac{1}{RC}\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & -\frac{1}{R}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \frac{1}{R}u$$

(*b*) The reachability and observability matrices becomes

$$\Gamma = \begin{bmatrix}B & AB\end{bmatrix} = \begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}, \qquad \Omega = \begin{bmatrix}C\\ CA\end{bmatrix} = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

which implies that the reachable and unobservable subspaces are

$$\mathcal{R} = \text{Im}(\Gamma) = \text{span}\left\{\begin{bmatrix}1\\ 1\end{bmatrix}\right\}$$

$$\overline{\mathcal{O}} = \text{Ker}(\Omega) = \text{span}\left\{\begin{bmatrix}1\\ 1\end{bmatrix}\right\}$$

Since $\mathcal{R} = \overline{\mathcal{O}}$ we make a change of coordinates such that

$$\mathbf{R}^2 = \mathcal{R} \cap \overline{\mathcal{O}} \oplus V_{\bar{r},o}$$

Here we pick $V_{\bar{r},o} = \text{span}\left\{\begin{bmatrix}1\\ -1\end{bmatrix}\right\}$. This gives the change of basis

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}z_{r,\bar{o}}\\ z_{\bar{r},o}\end{bmatrix}$$

A simple calculation gives the dynamics in the new coordinates

$$\begin{bmatrix}\dot{z}_{r,\bar{o}}\\ \dot{z}_{\bar{r},o}\end{bmatrix} = -\begin{bmatrix}z_{r,\bar{o}}\\ z_{\bar{r},o}\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}u$$

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}z_{r,\bar{o}}\\ z_{\bar{r},o}\end{bmatrix} + u$$

(*c*) We see that $z_{r,\bar{o}}$ can be controlled but it is not observable in the output. On the other hand, $z_{\bar{r},o}$ is observable in the output but is independent of the control. The reason is that the transfer function is equivalent to a pure resistance when $R = L = C = 1$

$$G(s) = \frac{y(s)}{u(s)} = 1.$$

2. (*a*) We can rewrite the transfer function as

$$G(s) = \frac{1}{\chi(s)}(N_0 + N_1 s) + D$$

where

$$\chi(s) = s^2 + 4s + 3$$
$$N_0 = \begin{bmatrix} 1 & 0 \\ -1 & -4 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

We immediately get

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 0 & -4 & 0 \\ 0 & -3 & 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -4 & 0 & -4 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

(*b*) By using the following partial fraction expansion and series expansions

$$\frac{1}{(s+1)(s+3)} = \frac{1}{2}\left(\frac{1}{s+1} - \frac{1}{s+3}\right)$$
$$\frac{1}{s+1} = \frac{1}{s}\frac{1}{1-(-s^{-1})} = \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} + \dots$$
$$\frac{1}{s+3} = \frac{1}{s}\frac{1}{1-(-3s^{-1})} = \frac{1}{s} - \frac{3}{s^2} + \frac{9}{s^3} + \dots$$

we get

$$G(s) = \frac{1}{s}R_1 + \frac{1}{s^2}R_2 + \dots + D$$

where

$$R_1 = \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1 & 0 \\ -1 & 12 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

This gives the standard observable realization

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 0 & -4 & 0 \\ 0 & -3 & 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & -4 \\ 1 & 0 \\ -1 & 12 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

(*c*) Let us first fix some notation for equivalence relations for matrices. We use the notations

$$A \sim_Q B \quad if \quad AQ = B$$
$$A \;{}_P{\sim}\; B \quad if \quad PA = B$$
$$A \;{}_P{\sim}_Q\; B \quad if \quad PAQ = B$$

To compute $n = \text{rank}(H_2)$ we do some elementary row operations. Let us use the standard controllable realization. We have ($R_k = CA^{k-1}B$)

$$H_2 = \begin{bmatrix} R_1 & R_2 \\ R_2 & R_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -4 & -1 & 12 \\ 1 & 0 & -4 & 0 \\ -1 & 12 & 4 & -36 \end{bmatrix}$$

$$_{P_1}\!\sim \begin{bmatrix} 1 & 0 & -4 & 0 \\ -1 & 12 & 4 & -36 \\ 0 & -4 & -1 & 12 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$_{P_2}\!\sim \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 12 & 0 & -36 \\ 0 & -4 & -1 & 12 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$_{P_3}\!\sim \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$_{P_4}\!\sim \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where

$$P_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \ P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/12 & 0 & 0 \\ 0 & 1/3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

From this we see that $n = \text{rank}(H_2) = 3$. We have shown

$$PH_2 = \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad P = P_4P_3P_2P_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1/12 & 1/12 \\ 0 & -1 & -1/3 & -1/3 \\ 1 & 1 & 1/3 & 1/3 \end{bmatrix}$$

If we continue with elementary column operations we get

$$\begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim_Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and thus

$$PH_2Q = \begin{bmatrix} I_3 & 0_{3,1} \\ 0_{1,3} & 0 \end{bmatrix}$$

(d) We follow Ho's algorithm

(1) $\deg(\chi(s)) = 2$

(2) From (c) we have $n = \text{rank}(H_2) = 3$.

(3) A minimal realization is obtained as

$$A = \begin{bmatrix} I_3 & 0_{3,1} \end{bmatrix} P\sigma(H_2)Q \begin{bmatrix} I_3 \\ 0_{1,3} \end{bmatrix} =$$

$$B = \begin{bmatrix} I_3 & 0_{3,1} \end{bmatrix} P \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

$$C = \begin{bmatrix} R_1 & R_2 \end{bmatrix} Q \begin{bmatrix} I_3 \\ 0_{1,3} \end{bmatrix}$$

where

$$\sigma(H_2) = \begin{bmatrix} R_2 & R_3 \\ R_3 & R_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 & 0 \\ -1 & 12 & 4 & -36 \\ -4 & 0 & 13 & 0 \\ 4 & -36 & -13 & 108 \end{bmatrix}$$

We get the minimal realization

$$A = \begin{bmatrix} -4 & 0 & -3 \\ 0 & -3 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -4 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

(Oa) We have (Ω_2 is sufficient because of Corollary 5.2.10 in Lindquist and Sand)

$$\Omega_2 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -4 & 0 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 12 & -1 & 12 \end{bmatrix} \sim_Q \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 12 & -1 & 0 \end{bmatrix}}_{\Omega_1}$$

where $Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. It follows that the realization is not observable and the unobservable space is

$$\bar{\mathcal{O}} = \text{Ker}(\Omega) = Q\text{Ker}(\Omega_1) = Q\text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(Ob) The reachability matrix is

$$\Gamma = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -4 & -1 & 12 \\ 1 & 0 & -4 & 0 \\ -1 & 12 & 4 & -36 \end{bmatrix}$$

By performing elementary row operations we get

$$\Gamma = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -4 & -1 & 12 \\ 1 & 0 & -4 & 0 \\ -1 & 12 & 4 & -36 \end{bmatrix} {}_{P_1}\!\sim \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -4 & -1 & 12 \\ 0 & 12 & 0 & -36 \\ -1 & 12 & 4 & -36 \end{bmatrix} {}_{P_2}\!\sim \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 12 & 0 & -36 \\ -1 & 12 & 4 & -36 \end{bmatrix}$$
$$ {}_{P_3}\!\sim \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 12 & 0 & -36 \\ -1 & 12 & 4 & -36 \end{bmatrix}}_{\Gamma_1}$$

or equivalently $P\Gamma = \Gamma_1$, where $P = P_3P_2P_1$ and

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It follows that

$$\mathcal{R} = \text{Im}(\Gamma) = P^{-1}\text{Im}(\Gamma_1) = P^{-1}\text{Im}\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 12 & 0 & -36 \\ -1 & 12 & 4 & -36 \end{bmatrix}\right)$$
$$= P_1^{-1}P_2^{-1}P_3^{-1}\text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right\}$$

where we used

$$P_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad P_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \\ 0 & 1 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \end{bmatrix}, \quad P_3^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. (*a*) (*i*) → (*ii*): We have

$$\begin{aligned}
P &= \sum_{k=0}^{\infty}(A^T)^k C^T C A^k \\
&= \sum_{k=0}^{n-1}(A^T)^k C^T C A^k + \sum_{k=n}^{\infty}(A^T)^k C^T C A^k \\
&= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^T \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} + \sum_{k=n}^{\infty}(A^T)^k C^T C A^k \\
&= \Omega^T\Omega + \sum_{k=n}^{\infty}(A^T)^k C^T C A^k
\end{aligned}$$

The first term is positive definite since $\text{Ker}(\Omega^T\Omega) = \text{Ker}(\Omega) = \{0\}$, which follows by the observability of (A, C). This shows that $P > 0$.

$(ii) \Rightarrow (i)$: Let $V(x) = x^T P x$. We have

$$V(x_{k+1}) - V(x_k) = x_k^T(A^T P A - P)x_k = -\|Cx_k\|^2$$

Summation from $k = 0$ to $k = N$ gives

$$\sum_{k=0}^{N} \|Cx_k\|^2 = V(x_0) - V(x_N) \leq V(x_0) < \infty$$

Hence, the sum on the left hand side must converge and this implies that $Cx_k \to 0$ as $k \to 0$. This in turn implies that as $k \to \infty$

$$x_k \to \left\{x : CA^k x = 0,\ \forall k = 0, 1, 2, \dots\right\} = \{0\}$$

where the last equality follows by the observability of the pair (A, C). This proves convergence and hence that A must be stable.