KTH Matematik

Homework 3
Mathematical Systems Theory, 5B1742
Spring 2006

1. Consider the dynamics

$$\dot{x} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

Find a state feedback that places the poles at $\{-1, -1, -2, -3\}$. You may not use **place** or any other special Matlab function.

2. The following stochastic system model distinguishes itself from the basic model in Lindquist and Sand because the same noise signal enters the state and the measurement equations.

$$\begin{cases} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases} \tag{1}$$

The Kalman filter theory can be extended to fit also this situation. The resulting Kalman gain becomes $K(t) = [AP(t)C' + BD'][CP(t)C' + DD']^{-1}$, where $P(t)$ is the solution to the matrix-Riccati-equation

$$P(t+1) = AP(t)A' - [AP(t)C' + BD'][CP(t)C' + DD']^{-1}[AP(t)C' + BD']' + BB'$$

with appropriate initialization. Consider the following stochastic model

$$\begin{cases} x(t+1) &= ax(t) + w(t) \\ y(t) &= x(t) + v(t) \end{cases} \tag{2}$$

whare all variables are scalar valued and

$$\mathrm{E} \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \begin{bmatrix} w(s) & v(s) \end{bmatrix} = \begin{bmatrix} m & ar \\ ar & r \end{bmatrix} \delta_{t,s}$$

(a) Determine the steady-state Kalman gain K_∞ corresponding to the above model and verify that it does not depend on m and r.

(b) Is the resulting Kalman filter a stable system?

(c) Express $x(t+1)$ as a function of $y(t), v(t)$ and $w(t)$, and motivate the earlier results.

3. Prove that the solution to the discrete time LQ problem

$$\min_u \underbrace{x_N^T S x_N + \sum_{k=0}^{N-1} (x_k^T Q x_k + u_k^T R u_k)}_{J(u)} \quad \text{subj. to} \quad \begin{cases} x_{k+1} = Ax_k + Bu_k \\ x_0 \text{ is given} \end{cases}$$

is the feedback control $u_k = -(R+B^T P_{k+1} B)^{-1} B^T P_{k+1} A x_k$ where P_k is the solution to the Riccati equation

$$\begin{aligned} P_k &= A^T P_{k+1} A + Q - A^T P_{k+1} B (R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A \\ P_N &= S \end{aligned}$$

Furthermore, show that the optimal cost is $\min_u J(u) = x_0^T P_0 x_0$.
We assume that $S = S^T \geq 0$, $Q = Q^T \geq 0$, and $R = R^T > 0$.

Hint: Use completion of squares to prove the result. In order to verify that the Riccati equation has a well defined solution you should verify that $P_k \geq 0$. One way to do this is to consider optimization from an arbitrary starting time $k = k_0$ to N, where $k_0 \in \{0, 1, \ldots, N-1\}$. This will show that P_k is positive semidefinite since the objective function (the cost) is postive.