KTH Matematik

HW3 in Mathematical Systems Theory, 2006
Answers and solution sketches

1. Let us first consider the upper left part of the system

$$\dot{x}_1 = A_1 x_1 + B_1 u_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u_1$$

We first transform to controllable canonical form. First determine t from the equation

$$t \begin{bmatrix} B_1 & A_1 B_1 & A_1^2 B_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad t = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$$

Then make a coordinate change with

$$T = \begin{bmatrix} t \\ tA_1 \\ tA_1^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

which gives the transformed system ($z_1 = T x_1$)

$$\dot{z}_1 = TAT^{-1} z_1 + TBu = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} z_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_1$$

The feedback $u_1 = g z_1$, where $g = \begin{bmatrix} -5 & -12 & -7 \end{bmatrix}$ results in the closed loop eigenvalues $\{-1, -2, -3\}$. In the original coordinates we should use $u_1 = k_1 x_1$ with $k_1 = g_1 T = \begin{bmatrix} -12 & -12 & 5 \end{bmatrix}$. For the full system we can use the feedback $u = Kx$ with

$$K = \begin{bmatrix} -12 & -12 & 5 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \tag{1}$$

2. (a) We can model the system as

$$\begin{aligned} x(t+1) &= ax(t) + bu(t) \\ y(t) &= x(t) + du(t) \end{aligned}$$

where $u(t)$ is a white noise process ($Eu(t) = 0$ and $Eu(t)u(s) = \delta_{ts}$) and $b = \sqrt{m}$, $d = \sqrt{r}$, and $\sqrt{m} = a\sqrt{r}$. We can now use the result stated in the problem. The stateionary solution is obtained by having $p(t+1) = p(t) = p$, which gives the ARE :

$$\begin{aligned} &p = a^2 p - (ap + l)(p + r)^{-1}(ap + l) + m, \quad \text{where } l = ar, \\ &p^2 + p(r(1 + a^2) - m) + r(a^2 r - m) = 0. \end{aligned}$$

It has two solutions, $p = -r$ and $p = m - ar^2$. The only valid one is the second because r can not be negative since it is a variance. Then we have $k_\infty = \frac{a(m - a^2 r) + ar}{m - a^2 r + r} = a$, which is independent of m and r.

(b) We have that $a - k_\infty = a - a = 0$, so the steady-state Kalman filter

$$\hat{x}(t+1) = a\hat{x}(t) + K_\infty(y(t) - \hat{x}(t)) = (a - K_\infty)\hat{x}(t) + K_\infty y(t) = ay(t)$$

The error dynamics becomes ($e(t) = x(t) - \hat{x}(t)$)

$$e(t+1) = ax(t) + w(t) - ay(t) = w(t) - av(t)$$

this is a stable discrete time system because the eigenvalue is at 0.

(c) We have that $x(t+1) = a(y(t) - w(t)) + v(t)$ and since the best estimates for the noises are $w(t) = v(t) = 0$, we have that $\hat{x}(t+1) = ay(t)$, which is what the result in a) says.

3. Define $V(x,k) = x_k^T P_k x_k$, which satisfies

$$V(x_{k+1}, k+1) - V(x_k, k) = (Ax_k + Bu_k)^T P_{k+1}(Ax_k + Bu_k) - x_k^T P_k x_k$$

By summing the above identity from $k = 0$ to $N-1$ we get

$$\begin{aligned}
J(u) - V(x_0, 0) &= x_N^T S x_N + \sum_{k=0}^{N-1}(x_k^T Q x_k + u_k^T R u_k) \\
&- V(x_N, N) + \sum_{k=0}^{N-1}(Ax_k + Bu_k)^T P_{k+1}(Ax_k + Bu_k) - x_k^T P_k x_k \\
&= x_N^T(S - P_N)x_N + \sum_{k=0}^{N-1} \|u_k + (R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A x_k\|^2_{(R+B^T P_{k+1} B)} \\
&\quad + x_k^T(-P_k + A^T P_{k+1} A + Q - A^T P_{k+1} B (R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A) x_k
\end{aligned}$$

where the last equality was obtained using completion of squares and where the norm notation should be interpreted as $\|u\|_Q^2 = x^T Q x$. By using the Riccati equation tis simplifies to

$$J(u) - V(x_0, 0) = \sum_{k=0}^{N-1} \|u_k + (R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A x_k\|^2_{(R+B^T P_{k+1} B)} \geq 0$$

with equality if $u_k = -(R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A x_k$.

To show that the Riccati equation is well defined we will prove that $P_k \geq 0$, for $k = 0, 1, \ldots, N$. This will ensure that the inverse of $R + B^T P_k B$ is well defined. The same arguments as above prove that

$$\begin{aligned}
&\underbrace{x_N^T S x_N + \sum_{k=k_0}^{N-1}(x_k^T Q x_k + u_k^T R u_k)}_{J(k_0,u)} - V(x_{k_0}, k_0) \\
&\qquad = \sum_{k=k_0}^{N-1} \|u_k + (R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A x_k\|^2_{(R+B^T P_{k+1} B)}
\end{aligned}$$

Hence,

$$V(x_{k_0}, k_0) = x_{k_0}^T P_{k_0} x_{k_0} = \min_u J(k_0, u) \geq 0$$

which implies that $P_{k_0} \geq 0$ since x_{k_0} is arbitrary.

This completes the proof.