

CHAPTER 2

Invariant and controlled invariant subspaces

In this chapter we introduce two important concepts: invariant subspace and controlled invariant subspace, which will be used later on to solve many control problems.

2.1. Invariant subspaces

Consider an n -dimensional linear system

$$(2.1) \quad \dot{x} = Ax$$

where $x \in \mathbb{R}^n$.

DEFINITION 2.1. *A set $\Omega \subseteq \mathbb{R}^n$ is called an invariant set of (2.1) if for any initial condition $x_0 \in \Omega$, we have $x(x_0, t) = e^{At}x_0 \in \Omega$, $\forall t \geq 0$.*

Some trivial examples of invariant sets are \mathbb{R}^n and $x = \{0\}$.

In this course we only consider a special class of invariant sets: invariant subspaces. Now let us discuss conditions for a subspace \mathcal{S} to be invariant. Since by Taylor expansion we have

$$x(x_0, t) = x_0 + tAx_0 + \frac{t^2}{2}A^2x_0 + \cdots,$$

it is obvious that if $A^i x_0 \in S \forall i \geq 0$, then $x(x_0, t) \in S$, $\forall t \geq 0$. Naturally this argument is true only if \mathcal{S} is a linear subspace. It is easy to see as a sufficient condition

$$(2.2) \quad Az \in \mathcal{S} \quad \forall z \in \mathcal{S}.$$

In other words, this condition implies that if we define a mapping from \mathbb{R}^n to \mathbb{R}^n : $w = Az$, then the image of $S \subseteq \mathbb{R}^n$ is contained in \mathcal{S} . We denote this by

$$(2.3) \quad A\mathcal{S} \subseteq \mathcal{S}.$$

Now we show this condition is also necessary for S to be invariant.

PROPOSITION 2.1. *A necessary and sufficient condition for a linear subspace \mathcal{S} to be invariant under (2.1) is that condition (2.3) holds.*

PROOF

We only show the necessity here. Suppose there exists a point $x_0 \in \mathcal{S}$ such that $Ax_0 \notin \mathcal{S}$. Then when t is sufficiently small, we have

$$x(x_0, t) = x_0 + tAx_0 + \mathcal{O}(t^2),$$

which does not belong to \mathcal{S} , since \mathcal{S} is closed. ■

EXAMPLE 2.1. Consider

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Show that $S = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ is invariant.

We first use the definition of invariant set to show this. It is easy to see that the set can be redefined as $S = \{x \in \mathbb{R}^2 : x_1 + x_2 = 0\}$. Then to show S to be invariant is to show $x_1(x_0, t) + x_2(x_0, t) = 0 \forall t \geq 0$ if $x_0 \in S$. This is equivalent to showing $\dot{x}_1 + \dot{x}_2 = 0$ for all $(x_1, x_2)^T \in S$. We have $\dot{x}_1 + \dot{x}_2 = -2x_2 - x_2 + x_1 = -(x_1 + x_2) = 0$ if $(x_1, x_2)^T \in S$.

We can also show this with Proposition 2.1, since $A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = - \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

REMARK 2.1. As an example to show the above result is only true for subspaces, we consider a circle defined by $R = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$. It is easy to show (as an exercise) that this set is invariant under the system

$$\begin{aligned} \dot{x}_1 &= \omega x_2 \\ \dot{x}_2 &= -\omega x_1, \end{aligned}$$

where ω is any positive number.

We ask the reader to check if $Az \subset R$ for any $z \in R$.

Then we can use condition (2.3) as an alternative definition for invariant subspace.

DEFINITION 2.2. A linear subspace \mathcal{S} is A -invariant (invariant under $\dot{x} = Ax$) iff $A\mathcal{S} \subseteq \mathcal{S}$.

2.2. Controlled invariant subspaces

Now we consider a control system

$$(2.4) \quad \dot{x} = Ax + Bu$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

DEFINITION 2.3. \mathcal{S} is called a controlled invariant subspace of (2.4) if there exists a feedback control $u = Fx$ such that \mathcal{S} is an invariant subspace of

$$\dot{x} = (A + BF)x.$$

Similar to invariant subspace, we can also give another equivalent definition.

DEFINITION 2.4. \mathcal{S} is an (A, B) -invariant (controlled invariant) subspace if there exists a matrix F such that

$$(2.5) \quad (A + BF)\mathcal{S} \subseteq \mathcal{S}.$$

Such an F is called a **friend** of \mathcal{S} .

We denote the set of friends by $\mathcal{F}(\mathcal{S})$. The following theorem provides a fundamental characterization of (A, B) -invariant subspaces that removes the explicit involvement of the feedback matrix F .

THEOREM 2.2. \mathcal{S} is (A, B) -invariant if and only if

$$(2.6) \quad A\mathcal{S} \subseteq \mathcal{S} + \text{Im } B.$$

PROOF

Necessity: Suppose F is a friend, then

$$(A + BF)\mathcal{S} \subseteq \mathcal{S}.$$

or

$$A\mathcal{S} \subseteq \mathcal{S} - B(F\mathcal{S}).$$

Since $B(F\mathcal{S}) \subseteq \text{Im } B$, thus (2.6) holds.

Sufficiency: The proof is constructive and is given as follows. ■

We now give an algorithm for finding a friend of \mathcal{V} , which also serves as a proof of the sufficiency of Theorem 2.2.

Algorithm for finding F

Let $\{v_1, v_2, \dots, v_r\}$ be a basis for \mathcal{V} . Since \mathcal{V} satisfies $A\mathcal{V} \subseteq \mathcal{V} + \text{Im } B$, there is for each $i = 1, \dots, r$ a $w_i \in \mathcal{V}$ and a $u_i \in \mathbb{R}^m$ such that

$$Av_i = w_i + Bu_i.$$

Let F be a $m \times n$ -matrix such that $Fv_i = -u_i$ for $i = 1, 2, \dots, r$ (if $r < n$ then F is not unique). Then $Av_i = w_i - BFv_i$, i.e., $(A + BF)v_i = w_i \in \mathcal{V}$ and therefore $(A + BF)\mathcal{V} \subseteq \mathcal{V}$.

EXAMPLE 2.2. Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Is the subspace $\mathcal{V} = \{x \in \mathbb{R}^2 : x_1 = x_2\}$ (A, B) -invariant? If so, find a friend of \mathcal{V} .

Clearly, \mathcal{V} is spanned by $v = [1 \ 1]'$. Since

$$Av = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathcal{V} + \text{Im } B,$$

\mathcal{V} is (A, B) -invariant, and we can let $u = 2$. To find F we must solve the under-determined system of equations $Fv = -u$, i.e.,

$$\begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2.$$

The set $\mathcal{F}(\mathcal{V})$ is the affine space $\{(\lambda - 2, -\lambda); \lambda \in \mathbb{R}\}$. Choose, e.g., $F = \begin{bmatrix} -2 & 0 \end{bmatrix}$.

Is the subspace $\mathcal{V} := \{x \in \mathbb{R}^2 : x_1 = 0\}$ (A, B) -invariant?

2.3. Reachability subspaces

In the rest of this chapter, we study the most elementary class of controlled invariant subspaces: *reachability (controllability) subspace*.

DEFINITION 2.5. *We use the notation $\langle A|S \rangle$ to denote the minimal A -invariant subspace that contains subspace S .*

Naturally if S is already A -invariant, then $\langle A|S \rangle = S$. $\langle A|S \rangle$ can be computed in the following way:

- (1) Let $S_0 = S$, check if $AS_0 \subseteq S_0$. If yes, stop. Otherwise,
- (2) Let $S_{k+1} = AS_k + S_k$, $k \geq 0$.
- (3) Check if $AS_{k+1} \subseteq S_{k+1}$. If yes, stop. Otherwise return to step 2.

Consider again (2.4). Recall that the reachable (controllable) subspace of (2.4) can be defined with our notation as

$$\langle A|\text{Im } B \rangle = \text{span}\{B, AB, \dots, A^{n-1}B\},$$

namely, the minimal A -invariant subspace that contains $\text{Im } B$. However, for many complex control problems, such as the problem of controllability under constraints discussed in the introduction, more refined study of reachability is needed.

Now consider the feedback law

$$(2.7) \quad u = Fx + Gv.$$

The corresponding closed-loop system

$$\dot{x} = (A + BF)x + BGv$$

has the reachable subspace

$$(2.8) \quad \mathcal{R} = \langle A + BF|\text{Im } BG \rangle.$$

REMARK 2.2. *By construction, \mathcal{R} is (A, B) -invariant.*

DEFINITION 2.6. *A subspace \mathcal{R} is called a reachability subspace of (2.4) if there are F and G such that (2.8) holds.*

EXAMPLE 2.3. *If $G = I$ then*

$$\mathcal{R} = \langle A + BF|\text{Im } B \rangle = \langle A|\text{Im } B \rangle,$$

is the reachable subspace. If $G = 0$ then $\mathcal{R} = 0$. For a SISO-system it is obvious that these are the only possible reachability subspaces.

We now proceed with the analysis of reachability subspaces. The first theorem shows that the matrix G can be removed from the characterization of \mathcal{R} at the price of an implicit characterization, which however is of great use.

THEOREM 2.3. *A subspace \mathcal{R} is a reachability subspace if and only if there is an F such that*

$$(2.9) \quad \mathcal{R} = \langle A + BF|\text{Im } B \cap \mathcal{R} \rangle.$$

PROOF

Necessity: Suppose \mathcal{R} is a reachability subspace, i.e.,

$$(2.10) \quad \mathcal{R} = \langle A + BF \mid \text{Im } BG \rangle$$

for some F and G . Then $\text{Im } BG \subseteq \mathcal{R}$ and $\text{Im } BG \subseteq \text{Im } B$, i.e.,

$$\text{Im } BG \subseteq \text{Im } B \cap \mathcal{R}.$$

Hence,

$$(2.11) \quad \mathcal{R} \subseteq \langle A + BF \mid \text{Im } B \cap \mathcal{R} \rangle.$$

But \mathcal{R} is (A, B) -invariant and therefore

$$(A + BF)^k \mathcal{R} \subseteq \mathcal{R} \text{ for } k \geq 1$$

and

$$(2.12) \quad \langle A + BF \mid \text{Im } B \cap \mathcal{R} \rangle \subseteq \mathcal{R}.$$

Now (2.9) follows from (2.11) and (2.12).

Sufficiency: Suppose that (2.9) holds. It is enough to show that there is a G such that $\text{Im } B \cap \mathcal{R} = \text{Im } BG$, since this will imply (2.10).

Let p_1, p_2, \dots, p_q be a basis for $\text{Im } B \cap \mathcal{R}$. Then there is a linearly independent set $\{u_1, u_2, \dots, u_q\}$ such that

$$p_i = Bu_i \quad i = 1, 2, \dots, q,$$

since if the u_i 's were linearly dependent then the p_i 's would be linearly dependent as well. If we let the input space be \mathbb{R}^m it holds that $q \leq \dim(\text{Im } B) \leq m$. Choose u_{q+1}, \dots, u_m such that $\{u_1, \dots, u_m\}$ is a basis for \mathbb{R}^m . We want

$$BG u_i = \begin{cases} p_i & i = 1, 2, \dots, q, \\ 0 & i = q + 1, \dots, m \end{cases}$$

which yields $\text{Im } BG = \text{Im } B \cap \mathcal{R}$, i.e.,

$$BG[u_1, \dots, u_m] = [p_1, \dots, p_q, 0, \dots, 0] = B[u_1, \dots, u_q, 0, \dots, 0].$$

This is achieved by

$$G := [u_1, \dots, u_q, 0, \dots, 0][u_1, u_2, \dots, u_m]^{-1}.$$

■

We know that the reachability subspace \mathcal{R} is (A, B) -invariant and it is obvious that F in

$$(2.13) \quad \mathcal{R} = \langle A + BF \mid \text{Im } B \cap \mathcal{R} \rangle$$

is a friend of \mathcal{R} , i.e., $F \in \mathcal{F}(\mathcal{R})$.

EXAMPLE 2.4. Consider

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= u_1 \\ \dot{x}_3 &= x_1 + u_2. \end{aligned}$$

We will show that $V = \text{span}\{e_1, e_2\}$ is a reachability subspace.

It is easy to compute that $F = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ is a friend of V , and $\text{Im}B \cap V = \text{span}\{e_2\}$. Then, it is easy to calculate that $(A + BF)e_2 = e_1$ and $(A + BF)e_1 = e_1$. Thus $\langle A + BF | \text{Im}B \cap V \rangle = V$. We note that $V_1 = \text{span}\{e_1\}$ is an (A, B) -invariant subspace but not a reachability subspace since $\text{Im}B \cap V_1 = 0$.

The next theorem shows that the representation (2.13) is independent of the actual choice of $F \in \mathcal{F}(\mathcal{R})$.

THEOREM 2.4. *Let \mathcal{R} be a reachability subspace and let $\hat{F} \in \mathcal{F}(\mathcal{R})$, i.e. an arbitrary friend of \mathcal{R} . Then*

$$\mathcal{R} = \langle A + B\hat{F} | \text{Im} B \cap \mathcal{R} \rangle.$$

PROOF

From Theorem 2.3 follows the existence of an F such that

$$\mathcal{R} = \langle A + BF | \text{Im} B \cap \mathcal{R} \rangle.$$

Now let $\hat{F} \in \mathcal{F}(\mathcal{R})$ be an arbitrary friend and form

$$\hat{\mathcal{R}} = \langle A + B\hat{F} | \text{Im} B \cap \mathcal{R} \rangle.$$

Since $(A + B\hat{F})\mathcal{R} \subseteq \mathcal{R}$, it holds that $\hat{\mathcal{R}} \subseteq \mathcal{R}$.

We shall show that $\mathcal{R} \subseteq \hat{\mathcal{R}}$ by induction. Clearly $\text{Im} B \cap \mathcal{R} \subseteq \hat{\mathcal{R}}$. Assume that

$$(A + BF)^k(\text{Im} B \cap \mathcal{R}) \subseteq \hat{\mathcal{R}}.$$

Then

$$\begin{aligned} (A + BF)^{k+1}(\text{Im} B \cap \mathcal{R}) &\subseteq (A + BF)\hat{\mathcal{R}} \\ &\subseteq (A + B\hat{F})\hat{\mathcal{R}} + B(F - \hat{F})\hat{\mathcal{R}} \\ &\subseteq \hat{\mathcal{R}} \text{ if } B(F - \hat{F})\hat{\mathcal{R}} \subseteq \hat{\mathcal{R}} \end{aligned}$$

If we can show the last inclusion, then it follows by induction that $\mathcal{R} \subseteq \hat{\mathcal{R}}$.

We need to introduce a lemma here in order to carry on the proof.

LEMMA 2.5. *Let $F_1 \in \mathcal{F}(\mathcal{V})$. Then $F_2 \in \mathcal{F}(\mathcal{V})$ if and only if $B(F_1 - F_2)\mathcal{V} \subseteq \mathcal{V}$.*

PROOF (Proof of the lemma)

(only if) Suppose $F_1, F_2 \in \mathcal{F}(\mathcal{V})$. Then for all $v \in \mathcal{V}$ it holds that $(A + BF_1)v \in \mathcal{V}$ and $(A + BF_2)v \in \mathcal{V}$, which implies that $B(F_1 - F_2)v \in \mathcal{V}$.

(if) Let $v \in \mathcal{V}$. Then $(A + BF_1)v + B(F_2 - F_1)v = (A + BF_2)v \in \mathcal{V}$, since the terms on the left hand side are in \mathcal{V} . \blacksquare

Now we return to the proof of the theorem. Since $\hat{\mathcal{R}} \subseteq \mathcal{R}$ it holds that

$$B(F - \hat{F})\hat{\mathcal{R}} \subseteq B(F - \hat{F})\mathcal{R} \subseteq \{\text{Lemma 2.5}\} \subseteq \mathcal{R}.$$

But $B(F - \hat{F})\hat{\mathcal{R}} \subseteq \text{Im } B$ and therefore

$$B(F - \hat{F})\hat{\mathcal{R}} \subseteq \text{Im } B \cap \mathcal{R} \subseteq \hat{\mathcal{R}}.$$

■

Combining Theorem 2.3 and Theorem 2.4 we obtain the following result, which can be used to test whether a given subspace is a reachability subspace.

COROLLARY 2.6. *Suppose \mathcal{V} is (A, B) -invariant and let $F \in \mathcal{F}(\mathcal{V})$ be an arbitrary friend of \mathcal{V} . The necessary and sufficient condition for \mathcal{V} to be a reachability subspace is that*

$$\langle A + BF | \text{Im } B \cap \mathcal{V} \rangle = \mathcal{V}.$$

2.4. Maximal reachability subspaces

Consider the class $S(\mathcal{Z})$ of (A, B) -invariant subspaces contained in \mathcal{Z} , and in particular reachability subspaces \mathcal{R} such that $\mathcal{R} \in S(\mathcal{Z})$. All these satisfy

$$(2.14) \quad \mathcal{R} \subseteq \mathcal{S}^*(\mathcal{Z}),$$

where $\mathcal{S}^*(\mathcal{Z})$ is the maximal (A, B) -invariant subspace in \mathcal{Z} . The existence of \mathcal{S}^* is shown as follows.

LEMMA 2.7. *Let \mathcal{Z} be a subspace of \mathbb{R}^n . Then, the class $S(\mathcal{Z})$ of all (A, B) -invariant subspaces $\mathcal{S} \subseteq \mathcal{Z}$ has a maximal element $\mathcal{S}^*(\mathcal{Z})$ in the sense that*

$$\mathcal{S} \subseteq \mathcal{S}^*(\mathcal{Z}) \text{ for all } \mathcal{S} \in S(\mathcal{Z}).$$

PROOF

Note first that the set $S(\mathcal{Z})$ is closed under addition, i.e., if $\mathcal{S}_1, \mathcal{S}_2 \in S(\mathcal{Z})$, then $\mathcal{S}_1 + \mathcal{S}_2 \subseteq \mathcal{Z}$ and

$$A(\mathcal{S}_1 + \mathcal{S}_2) = A\mathcal{S}_1 + A\mathcal{S}_2 \subseteq \mathcal{S}_1 + \mathcal{S}_2 + \text{Im } B.$$

Hence, $\mathcal{S}_1 + \mathcal{S}_2 \in S(\mathcal{Z})$.

Since \mathcal{Z} is of finite dimension, there is an element $\mathcal{S}^* \in S(\mathcal{Z})$ of largest dimension. If $\mathcal{S} \in S(\mathcal{Z})$, then $\mathcal{S} + \mathcal{S}^* \in S(\mathcal{Z})$ and $\mathcal{S}^* \subseteq \mathcal{S} + \mathcal{S}^*$. However, \mathcal{S}^* has maximal dimension and therefore, $\dim(\mathcal{S} + \mathcal{S}^*) = \dim \mathcal{S}^*$, and then, $\mathcal{S}^* = \mathcal{S} + \mathcal{S}^*$, that is, $\mathcal{S} \subseteq \mathcal{S}^*$. Thus, \mathcal{S}^* is maximal in terms of subspace inclusion. ■

Is there also a *maximal* \mathcal{R} that satisfies (2.14)? *Maximal* in the sense that it contains all other such reachability subspaces.

THEOREM 2.8. *Let \mathcal{S}^* be the maximal (A, B) -invariant subspace in \mathcal{Z} , and let $F \in \mathcal{F}(\mathcal{S}^*)$. Then the maximal reachability subspace in \mathcal{Z} is*

$$(2.15) \quad \mathcal{R}^* := \langle A + BF | \text{Im } B \cap \mathcal{S}^* \rangle.$$

Moreover, $F \in \mathcal{F}(\mathcal{R}^*)$, i.e.,

$$\mathcal{F}(\mathcal{R}^*) \supseteq \mathcal{F}(\mathcal{S}^*).$$

We develop the proof with the help of the following two lemmas. The first of them is a refinement of Theorem 2.4, where we learn that a reachability subspace can in fact be characterized by any friend of the smaller class of friends of an (A, B) -invariant subspace which generates the reachability subspace in a specific way.

LEMMA 2.9. *Let \mathcal{S} be (A, B) -invariant and let*

$$\mathcal{R} := \langle A + BF | \hat{\mathcal{B}} \rangle,$$

where $F \in \mathcal{F}(\mathcal{S})$ and $\hat{\mathcal{B}} = \text{Im } B \cap \mathcal{S}$. If \hat{F} is any matrix such that $B(\hat{F} - F)\mathcal{S} \subseteq \mathcal{S}$ then we also have that $\mathcal{R} = \langle A + B\hat{F} | \hat{\mathcal{B}} \rangle$.

REMARK 2.3. *Recalling Lemma 2.5 we see that the condition for \hat{F} in the above lemma amounts to $\hat{F} \in \mathcal{F}(\mathcal{S})$.*

PROOF

Let

$$\hat{\mathcal{R}} := \langle A + B\hat{F} | \hat{\mathcal{B}} \rangle$$

and

$$\mathcal{S}_i := \hat{\mathcal{B}} + (A + BF)\hat{\mathcal{B}} + \dots + (A + BF)^{i-1}\hat{\mathcal{B}}.$$

Then $\mathcal{S}_1 \subseteq \hat{\mathcal{R}}$.

Proceeding by induction, assume that $\mathcal{S}_i \subseteq \hat{\mathcal{R}}$. Then

$$\mathcal{S}_{i+1} = \hat{\mathcal{B}} + (A + BF)\mathcal{S}_i \subseteq \hat{\mathcal{B}} + (A + B\hat{F})\mathcal{S}_i + B(F - \hat{F})\mathcal{S}_i,$$

which is included in $\hat{\mathcal{R}}$ if

$$(2.16) \quad B(F - \hat{F})\hat{\mathcal{R}} \subseteq \hat{\mathcal{R}}.$$

If so, $\mathcal{R} = \mathcal{S}_n \subseteq \hat{\mathcal{R}}$ by induction.

We now show (2.16). Since $\hat{F} \in \mathcal{F}(\mathcal{S})$ and $\hat{\mathcal{B}} \subseteq \mathcal{S}$ it follows that $\hat{\mathcal{R}} \subseteq \mathcal{S}$. Therefore,

$$B(F - \hat{F})\hat{\mathcal{R}} \subseteq B(F - \hat{F})\mathcal{S} \subseteq \hat{\mathcal{B}} \subseteq \hat{\mathcal{R}}$$

and (2.16) follows. We have thus shown that $\mathcal{R} \subseteq \hat{\mathcal{R}}$.

If we interchange F and \hat{F} in the calculations above, we get $\hat{\mathcal{R}} \subseteq \mathcal{R}$. ■

LEMMA 2.10. *Let \mathcal{R} and \mathcal{S} be (A, B) -invariant, and suppose that $\mathcal{R} \subseteq \mathcal{S}$. Then, if $\hat{F} \in \mathcal{F}(\mathcal{R})$, there is an $F \in \mathcal{F}(\mathcal{R}) \cap \mathcal{F}(\mathcal{S})$ such that $F|_{\mathcal{R}} = \hat{F}|_{\mathcal{R}}$.*

PROOF

Let \mathcal{W} be a subspace such that

$$\mathcal{R} \oplus \mathcal{W} = \mathcal{S}.$$

Let $\{w_1, \dots, w_q\}$ be a basis for \mathcal{W} . Since \mathcal{S} is (A, B) -invariant and $\mathcal{W} \subseteq \mathcal{S}$, we have

$$Aw_i = v_i + Bu_i$$

for some $v_i \in \mathcal{S}$ and $u_i \in \mathbb{R}^m$. Now let $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be such that $Fx = \hat{F}x$ for $x \in \mathcal{R}$ and $Fw_i = -u_i$. Then $F|_{\mathcal{R}} = \hat{F}|_{\mathcal{R}}$ and $(A + BF)\mathcal{R} \subseteq \mathcal{R} \subseteq \mathcal{S}$. Moreover, $(A + BF)w_i = v_i$, i.e., $(A + BF)\mathcal{W} \subseteq \mathcal{S}$. Hence, $(A + BF)\mathcal{S} \subseteq \mathcal{S}$. \blacksquare

We now prove Theorem 2.8.

PROOF

We need to show that \mathcal{R}^* as defined by $\langle A + BF | \text{Im } B \cap \mathcal{S}^* \rangle$ where F is any friend of \mathcal{S}^* , is a reachability subspace in \mathcal{Z} , and moreover that it is maximal.

Since $\text{Im } B \cap \mathcal{S}^* \subseteq \mathcal{S}^*$ we have

$$\mathcal{R}^* \subseteq \langle A + BF | \mathcal{S}^* \rangle = \mathcal{S}^* \subseteq \mathcal{Z}$$

and we can always choose G such that $\text{Im } BG = \text{Im } B \cap \mathcal{S}^*$. So \mathcal{R}^* is a reachability subspace in \mathcal{Z} .

Next we show that $\mathcal{R} \subset \mathcal{R}^*$ for all reachability subspaces contained in \mathcal{Z} . If \mathcal{R} is an arbitrary reachability subspace in \mathcal{Z} , it can be expressed as

$$\mathcal{R} = \langle A + BF_0 | \text{Im } B \cap \mathcal{R} \rangle$$

for some $F_0 \in \mathcal{F}(\mathcal{R})$. Clearly, $\mathcal{R} \subseteq \mathcal{S}^*$. Moreover, by Lemma 2.10 there is an $F_1 \in \mathcal{F}(\mathcal{S}^*)$ such that

$$(2.17) \quad F_1|_{\mathcal{R}} = F_0|_{\mathcal{R}}.$$

Now if $x \in \mathcal{S}^*$ then

$$B(F - F_1)x = (A + BF)x - (A + BF_1)x \in \mathcal{S}^*,$$

since $F, F_1 \in \mathcal{F}(\mathcal{S}^*)$. Hence,

$$(2.18) \quad B(F - F_1)\mathcal{S}^* \subseteq \text{Im } B \cap \mathcal{S}^*.$$

Consequently,

$$(2.19) \quad \mathcal{R} = \langle A + BF_0 | \text{Im } B \cap \mathcal{R} \rangle$$

$$(2.20) \quad = \langle A + BF_1 | \text{Im } B \cap \mathcal{R} \rangle$$

$$(2.21) \quad \subseteq \langle A + BF_1 | \text{Im } B \cap \mathcal{S}^* \rangle$$

$$(2.22) \quad = \langle A + BF | \text{Im } B \cap \mathcal{S}^* \rangle = \mathcal{R}^*,$$

where (2.20) follows from (2.17), (2.21) follows from $\mathcal{R} \subseteq \mathcal{S}^*$, and (2.22) follows from Lemma 2.9 and (2.18). But \mathcal{R} is arbitrary, and therefore \mathcal{R}^* is the unique maximal reachability subspace in \mathcal{Z} . \blacksquare

We conclude this section with an example.

EXAMPLE 2.5. Compute \mathcal{R}^* contained in $\mathcal{Z} = \ker C$ for

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \text{ and } C = [1 \ 0 \ 0].$$

For this purpose, we need to compute \mathcal{S}^* first. Set $\mathcal{S}_0 = \ker C$, i.e.,

$$S_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and we check if

$$AS_0 \subset S_0 + \text{Im}B.$$

Since

$$[S_0, B] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

has full dimension, clearly,

$$AS_0 \subset S_0 + \text{Im}B.$$

Hence,

$$\mathcal{S}^* = \ker C = \text{Im} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = [v_1, v_2].$$

(What should we do if $S_0 \neq \mathcal{S}^*$?) The next step is to determine a friend for \mathcal{S}^* . Since $(A + BF)\mathcal{S}^* \subseteq \mathcal{S}^*$ implies

$$A\mathcal{S}^* \subseteq \mathcal{S}^* + B(-F)\mathcal{S}^*,$$

Therefore, we form

$$(2.23) \quad A[v_1, v_2] = \begin{bmatrix} 2 & 0 \\ 3 & 0 \\ 2 & 3 \end{bmatrix}$$

$$(2.24) \quad = \begin{bmatrix} 0 & 0 \\ 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}.$$

The first term in (2.24) is in \mathcal{S}^* and the second term has the form $B[u_1, u_2]$. Hence, to find F we must solve the system

$$-F[v_1, v_2] = [u_1, u_2],$$

i.e.,

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = - \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \text{ with solution } \begin{bmatrix} f_{11} & 0 & 0 \\ f_{21} & -2 & 0 \end{bmatrix}.$$

If we choose $f_{11} = f_{21} = 0$ then $A + BF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Finally, straight-

forward computations yields

$$\text{Im } B \cap \mathcal{S}^* = \text{Im} \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = \mathcal{R}^*.$$

2.5. Reachability under state constraints

Consider the system

$$(2.25) \quad \dot{x} = Ax + Bu$$

In this section we shall answer the following question. Let \mathcal{Z} be an arbitrary subspace of \mathbb{R}^n . Which states can be reached from the origin if we require that the trajectory lies in \mathcal{Z} ?

Before giving the main result we state some lemmas.

LEMMA 2.11. *Let $x(t, u)$ be the solution of a controlled differential equation and let \mathcal{M} be a subspace of \mathbb{R}^n . If $x(t, u) \in \mathcal{M}$ for all t then $\dot{x}(t, u) \in \mathcal{M}$ for all t .*

The proof is left as an exercise for the reader.

LEMMA 2.12. *Consider the system (2.25) and let \mathcal{Z} be a subspace of \mathbb{R}^n . If $x(t) \in \mathcal{Z}$ for $t \geq 0$ then $x(t) \in \mathcal{S}^*(\mathcal{Z})$ for $t \geq 0$.*

The proof is left as an exercise for the reader.

LEMMA 2.13. *Consider the system (2.25) and let \mathcal{Z} be a subspace of \mathbb{R}^n . If $x(0) = 0$ and $x(t) \in \mathcal{Z}$ for $t \geq 0$ then $x(t) \in \mathcal{R}^*(\mathcal{Z})$ for $t \geq 0$.*

PROOF

By Lemma 2.12 we know that $x(t) \in \mathcal{S}^*(\mathcal{Z})$ for $t \geq 0$. Now, let F be a friend of $\mathcal{S}^*(\mathcal{Z})$ and write the input as

$$u = Fx + v.$$

Then

$$Bv(t) = \dot{x}(t) - (A + BF)x(t) \in \mathcal{S}^*(\mathcal{Z}) \text{ for } t \geq 0,$$

by Lemma 2.11 and Lemma 2.12. Hence,

$$Bv(t) \in \text{Im } B \cap \mathcal{S}^*(\mathcal{Z}),$$

which implies that

$$x(t) = \int_0^t e^{(A+BF)(t-s)} Bv(s) ds \in \langle A + BF | \text{Im } B \cap \mathcal{S}^*(\mathcal{Z}) \rangle = \mathcal{R}^*(\mathcal{Z}).$$

■

We can now solve the problem of reachability under constraints.

THEOREM 2.14. *Let \mathcal{S} be the set of states that can be reached from the origin with trajectories in \mathcal{Z} , i.e.,*

$$(2.26) \quad \mathcal{S} := \{x \in \mathcal{Z} \mid \exists t_1 : x(t_1) = x \text{ and } x(t) \in \mathcal{Z} \forall t \in [0, t_1]\}.$$

Then,

$$\mathcal{R}^*(\mathcal{Z}) = \mathcal{S}.$$

PROOF

We first show that $\mathcal{S} \subseteq \mathcal{R}^*(\mathcal{Z})$. Let $x \in \mathcal{S}$. Then there is an input and a time t_1 such that $x(t) \in \mathcal{Z}$ and $x(t_1) = x$. By Lemma 2.13 it follows that $x(t) \in \mathcal{R}^*(\mathcal{Z})$ for $t \leq t_1$, and in particular, $x(t_1) \in \mathcal{R}^*(\mathcal{Z})$.

Next we show that $\mathcal{R}^*(\mathcal{Z}) \subseteq \mathcal{S}$. Let F and G be such that $\mathcal{R}^*(\mathcal{Z})$ is the reachable subspace of the closed-loop system

$$\dot{x} = (A + BF)x + BGv.$$

Clearly, all points in $\mathcal{R}^*(\mathcal{Z})$ can be reached by trajectories in $\mathcal{R}^*(\mathcal{Z})$. Hence, $\mathcal{R}^*(\mathcal{Z}) \subseteq \mathcal{S}$. ■