CHAPTER 8

Nonlinear systems

8.1. Introduction

A nonlinear control system can be generally expressed as follows:

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned}$$

where $x \in M \subset R^n$ is the state variable, $u \in R^m$ the input, and $y \in R^p$ the output.

In this course we focus on the so-called *affine control systems*:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{8.1}$$

Here we use two examples to illustrate why sometimes we have to deal with nonlinear control systems.

EXAMPLE 8.1 (Car steering system). *Here is a simplified model of a car:*

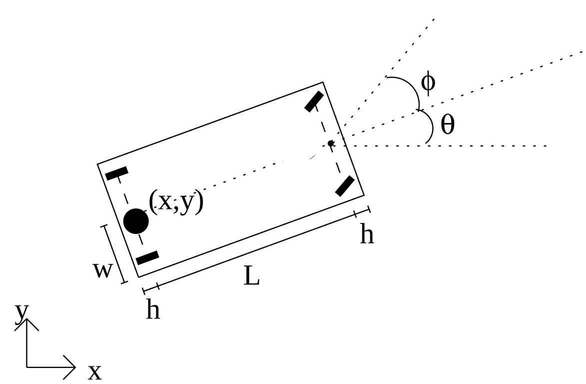

FIGURE 1. The geometry of the car-like robot, with position (x, y), orientation θ and steering angle ϕ.

One can easily write down the equations as:

$$\begin{aligned}\dot{x} &= v\cos(\theta)\\ \dot{y} &= v\sin(\theta)\\ \dot{\theta} &= \frac{v}{L}\tan\phi\end{aligned}$$

where x and y are Cartesian coordinates of the middle point on the rear axle, θ is orientation angle, v is the longitudinal velocity measured at that point, L is the distance of the two axles, and ϕ is the steering angle.

As we have mentioned in the introduction, if we consider v and $\frac{v}{L}\tan\phi$ as the two controls, then the linearization of the system is not controllable, while the nonlinear system itself is.

EXAMPLE 8.2 (Adaptive control). *Consider a one-dimensional linear system:*

$$\dot{x} = ax + u,$$

where the constant a is positive and unknown. We do not know the upper bound for a. In this case no linear control can guarantee the stability. However it is known that the following adaptive control

$$\begin{aligned}u &= -kx\\ \dot{k} &= x^2\end{aligned}$$

always stabilizes the system.

8.2. Controllability

DEFINITION 8.1. *A control system* (8.1) *is called* controllable *if for any two points x_1, x_2 in R^n, there exist a finite time T and an admissible control u such that $x(x_1, T, 0, u) = x_2$, where $x(x_0, t, t_0, u)$ denotes the solution of* (8.1) *at time t with initial condition x_1, initial time t_0 and control $u(\cdot)$.*

For a linear system

$$\dot{x} = Ax + Bu \quad x \in R^n \; u \in R^m$$

it is controllable if and only if the linear space

$$R = Im(B \; AB \; \cdots \; A^{n-1}B)$$

has dimension n. The simplest way to study controllability of a nonlinear system is to consider its linearization.

PROPOSITION 8.1. *Consider system* (8.1) *and x_0 where $f(x_0) = 0$. If the linearization at x_0 and $u = 0$*

$$\dot{z} = \frac{\partial f}{\partial x}(x_0)z + g(x_0)v \quad z \in R^n \; v \in R^m$$

is controllable, then the set of points that can be reached from x_0 in any finite time contains a neighborhood of x_0.

However, if we linearize the system in Example 8.1 around the origin, we will see the linearization is not controllable at all.

Now the question is, whether the nonlinear system itself is controllable?

8.2.1. Some Mathematical Preparations.

Manifold:

Suppose N is an open set in R^n. The set M is defined as

$$M = \{x \in N : \lambda_i(x) = 0, i = 1, \ldots, n-m\}$$

where λ_i are smooth functions.

If rank $\begin{bmatrix} \frac{\partial \lambda_1}{\partial x} \\ \vdots \\ \frac{\partial \lambda_{n-m}}{\partial x} \end{bmatrix} = n - m \ \forall x \in M$, then M is a (hyper)surface (which is a smooth manifold of dimension m).

Tangent vector and Tangent space:

We have all learned about tangent vectors and we know that the tangent space is just the collection of all the tangent vectors. Now we try to define tangent vector from a different angle.

It is known that the tangent space to R^n can be identified with R^n. Take a tangent vector $b \in R^n$ and smooth function $\lambda : R^n \to R$, then at any point $x \in R^n$, the rate of change of $\lambda(x)$ along the direction of b is

$$L_b\lambda := \lim_{\epsilon \to 0} \frac{1}{\epsilon}(\lambda(x + \epsilon b) - \lambda(x))$$

By Taylor expansion we have

$$L_b\lambda = \frac{\partial \lambda(x)}{\partial x} b = (\sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i})\lambda(x)$$

So we can also write a tangent vector b to R^n in the operator form:

$$b = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}$$

We see from this that

$$\left\{ \frac{\partial}{\partial x_i} \right\} \ i = 1, \ldots, n$$

is a basis for the tangent space to R^n.

Tangent space to a manifold M at point p (denoted by T_pM) can be defined similarly. However, we will not give a precise definition here. Interested reader can refer to [**11**].

Vector fields, Lie brackets:

Definition 8.2. *A* vector field f *on a smooth manifold* M *is a mapping assigning to each point* $p \in M$ *a tangent vector* $f(p) \in T_pM$*. A vector field*

f is smooth *over R^n (where $M = R^n$) if there exists n real-valued smooth functions $f_1, \ldots, f_n$ defined on R^n such that for all $q \in R^n$*

$$f(q) = \sum_{i=1}^{n} f_i(q) \frac{\partial}{\partial x_i},$$

where $x_1, \ldots, x_n$ form a basis for R^n.

DEFINITION 8.3. *Let λ be a smooth real-valued function on M. The* Lie derivative *of λ along f is a function $M \to R$, written $L_f\lambda$ and defined as*

$$(L_f\lambda)(p) := f(p)(\lambda).$$

When $M = R^n$ (and in general locally), it is represented by

$$L_f\lambda(p) = \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i}(p) f_i$$

Furthermore, we have

$$L_f\lambda(p) = \lim_{h \to 0} \frac{\lambda(\Phi_h^f(p)) - \lambda(p)}{h},$$

where $\Phi_h^f(p)$ denotes the solution to $\dot{x} = f(x)$ at $t = h$ and with the initial condition p.

Conventionally we denote

$$L_f L_g \lambda := L_f(L_g\lambda)$$

and

$$L_f^n \lambda := L_f(L_f^{(n-1)}\lambda)$$

For any two vector fields f and g on M, we define a new vector field, denoted by $[f, g]$, called the *Lie bracket* of the two vector fields, according to the rule:

$$[f, g](\lambda) := (L_f L_g \lambda) - (L_g L_f \lambda)$$

In local coordinates the expression of $[f, g]$ is given as

$$\frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

LEMMA 8.2. *The collection of all (smooth) vector fields on M (denoted as $V(M)$) with the product $[\cdot, \cdot]$ is a Lie algebra, i.e., $[\cdot, \cdot]$ has the following properties:*
1) it is skew commutative:

$$[f, g] = -[g, f]$$

2) it is bilinear over R:

$$[a_1 f_1 + a_2 f_2, g] = a_1[f_1, g] + a_2[f_2, g]$$

3) it satisfies the Jacobi identity:

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0.$$

Two vector fields f and g are called *commuting* if $\Phi_t^f \circ \Phi_s^g = \Phi_s^g \circ \Phi_t^f$, where Φ_t^f denotes the solution to $\dot{x} = f(x)$ at time t.

LEMMA 8.3. *f and g are commuting if and only if $[f, g] = 0$.*

As the dual to vector fields, now we study *one-forms* (covector fields). Recalling that T_pM is the tangent space to M at p, now we denote T_p^*M the dual space of T_pM, called the *cotangent space* to M at p. Elements of the cotangent space are called *cotangent vectors*. The dual basis is denoted by $dx_1|_p, \ldots, dx_n|_p$, defined by

$$dx_i|_p(\frac{\partial}{\partial x_j})|_p = \delta_{ij},\ i, j = 1, \ldots, n$$

Distributions

DEFINITION 8.4. *A* distribution *D on a manifold M is a map which assigns to each $p \in M$ a vector subspace $D(p)$ of the tangent space T_pM. D is called* smooth *if for each $p \in M$ there exists a neighborhood U of p and a set of smooth vector fields $f_i,\ i \in I$, such that for all $q \in U$*

$$D(q) = span\{f_i(q),\ i \in I\}.$$

Throughout the course we always assume a distribution is smooth and the index set I is finite.

A distribution is called *nonsingular* if for each $p \in M$ $dim(D(p))$ is the same.

A distribution D is called *involutive* if $[f, g] \in D$ whenever f, g are vector fields in D.

A manifold P is called an *integral manifold* of a distribution D if for each $p \in P$

$$T_pP = D(p)$$

8.2.2. Nonlinear Controllability and Accessibility. Now we return to the nonlinear control system

$$\dot{x} = f(x) + g(x)u \qquad x \in N \in R^n$$

where $g(x) = (g_1(x), \ldots, g_m(x))$.

A distribution $\Delta(x)$ is said to be invariant under vector field $f(x)$, if $\forall k(x) \in \Delta(x)$,

$$[f, k] \in \Delta(x).$$

DEFINITION 8.5 (Strong accessibility distribution R_c). *R_c is the smallest distribution which contains $span\{g_1, \ldots, g_m\}$ and is invariant under vector fields $f, g_1, \ldots, g_m$ and is denoted by*

$$R_c(x) =< f, g_1, \ldots, g_m | span\{g_1, \ldots, g_m\} > .$$

REMARK 8.1. *For linear systems, the strong accessibility distribution is*

$$R_c(x) =< Ax, b_1, \dots, b_m | ImB > .$$

Since we get the b_i-invariance (for $i = 1, \dots, m$) for free ($[b_i, b_j] = 0$) and

$$[b, Ax] = Ab$$

for any constant vector b, we have

$$R_c(x) =< A|ImB >,$$

which is the controllable subspace we studied in Chapter 2.

For nonlinear systems, it is in general very difficult to determine the controllability except for some special cases. Thus it is useful to study the so called *accessibility.*

PROPOSITION 8.4. *If at a point x_0, $dim(R_c(x_0)) = n$, then the system is locally strongly accessible from x_0. Namely, for any neighborhood of x_0, the set of reachable points at time T contains a non-empty open set for any $T > 0$ sufficiently small.*

A proof for this result and the one that follows is beyond the scope of this course. Interested readers may consult [**11**] for references.

PROPOSITION 8.5. *If $f = 0$, then $dim(R_c(x)) = n \quad \forall x \in N$ implies the system is controllable.*

Now we go back to the car example. With $u_1 = v$ and $u_2 = \frac{v}{L} \tan \phi$, we obtain that $R_c(x)$ has dimension 3 everywhere. Since $f = 0$ in this case, by Proposition 8.5 we know the system is controllable.

8.3. Stability of nonlinear systems

Consider a so-called autonomous system (where the time t does not appear in the right hand side of the equation)

$$\dot{x} = f(x). \tag{8.2}$$

Suppose $x = 0$ is an equilibrium (i.e. $f(0) = 0$) and the system has a unique solution for each initial condition in the domain of interest.

Without loss of generality, we always assume that we start at $t = 0$, namely $t_0 = 0$.

DEFINITION 8.6 (Stability concepts).

- **Stable :** $x = 0$ *is stable if* $\forall\ \epsilon > 0$, $\exists\ \delta(\epsilon) > 0$, *such that*

$$\|x_0\| < \delta(\varepsilon) \Rightarrow \|x(x_0, t)\| < \varepsilon \quad \forall t \geq 0.$$

- **Unstable :** $x = 0$ *is not stable.*
- **Attractive :** $x = 0$ *is attractive if*

$$\|x_0\| < \eta \Rightarrow \lim_{t \to \infty} x(x_0, t) = 0.$$

- **Asymptotically stable (a.s.):** *$x = 0$ is stable and attractive.*
- **Exponentially stable :** *there exist $k > 0,\ r > 0$, such that*

$$\|x(x_0, t)\| < k\|x_0\|e^{-rt} \qquad \forall t \geq 0,\ x_0 \in N(0).$$

EXAMPLE 8.3. *Consider*

$$\dot{x} = ax^n,$$

where $a \neq 0$. Then by solving the equation, we can easily conclude that

(1) *$x = 0$ is asymptotically stable if $a < 0$ and n is odd,*
(2) *$x = 0$ is unstable otherwise.*

8.3.1. Some Results on Stability of Linear Systems. Consider first a time-invariant system:

$$\dot{x} = Ax$$

Fact 1: $x = 0$ is asymptotically stable iff all the eigenvalues of A have negative real parts.

Fact 2: For linear time invariant systems, asymptotic stability is equivalent to exponential stability.

Fact 3: $x = 0$ is stable iff it does not have eigenvalues with positive real parts and for imaginary eigenvalues (including 0), their *algebraic multiplicity* should be equal to their *geometric multiplicity*[1].

Fact 4: if A is asymptotically stable, then and only then $\forall N < 0$, there exists a $P > 0$ such that

$$A^T P + PA = N.$$

In other words, if we take $V = x^T Px$, then

$$\dot{V} = x^T Nx < 0.$$

8.3.2. Stability of time-varying linear systems. Now we recall some results on the time varying case. We first use the following example to show that in general stability of such systems can not be decided only by eigenvalues of the matrix.

EXAMPLE 8.4. *Consider*

$$\dot{x} = A(t)x,$$

where $A(t) = P^{-1}(t)A_0P(t)$, and

$$A_0 = \begin{bmatrix} -1 & -5 \\ 0 & -1 \end{bmatrix},\ P(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}.$$

[1]Give an $n \times n$ matrix A, the algebraic multiplicity of an eigenvalue λ_0 is the number of times that λ_0 appears as a root of the characteristic polynomial $\rho_A(\lambda)$. The geometric multiplicity is the number of linearly independent eigenvectors (or the number of Jordan blocks) associated with λ_0.

Obviously, $A(t)$ has both eigenvalues -1 fore every t. But the solution to the system is $x(t) = \Psi(t)\Psi^{-1}(t_0)x(t_0)$, where

$$\Psi(t) = \begin{bmatrix} e^t(\cos(t) - \frac{1}{2}\sin(t)) & e^{-3t}(\cos(t) - \frac{1}{2}\sin(t)) \\ e^t(\sin(t) - \frac{1}{2}\cos(t)) & e^{-3t}(\sin(t) + \frac{1}{2}\cos(t)) \end{bmatrix}.$$

THEOREM 8.6. *Suppose $\dot{z} = A(t)z$ is exponentially stable. Then there exists an $\varepsilon > 0$ s.t. if $\|B(t)\| < \varepsilon, \quad \forall t \geq 0$, then $\dot{z}(t) = (A(t) + B(t))z(t)$ is also exponentially stable.*

THEOREM 8.7. *Suppose $\|A(t)\| < M, \quad \forall t \geq 0$ and the eigenvalues $\lambda_i(t)$ of $A(t)$ satisfy*

$$\mathcal{R}\mathrm{e}\,\{\lambda_i(t)\} \leq -r < 0, \quad \forall t \geq 0.$$

Then there exists an $\varepsilon > 0$ such that if $\|\dot{A}\| < \varepsilon, \quad \forall t \geq 0$, then $x = 0$ of

$$\dot{x} = A(t)x$$

is exponentially stable.

8.3.3. Principle of Stability in the first Approximation. Suppose (8.2) can be written as:

$$\dot{x} = Ax + g(x), \tag{8.3}$$

where $g(x) = o(\|x\|), \quad \forall x \in B_r, \quad \forall t \geq t_0 \geq 0$. Then we call

$$\dot{z} = Az, \tag{8.4}$$

the linearized system of (8.3).

THEOREM 8.8. *If the equilibrium $z = 0$ of (8.4) is exponentially stable, the $x = 0$ of (8.3) is also exponentially stable.*

THEOREM 8.9. *If A is constant matrix, and at least one of the eigenvalues is located in the open right half plane, then $x = 0$ of (8.3) is unstable.*

REMARK 8.2. *(Linear vs. nonlinear systems)*

1. *The principle of stability in the first approximation only applies to local stability analysis.*
2. *In the case where the linearized system is autonomous, and some of the eigenvalues are on the imaginary axis, and the rest are on the left open half plane, ("the critical case"), then one has to consider the nonlinear terms to determine stability.*
3. *In the case of a time-varying linearized system, if $z = 0$ is only (not uniformly–which we did not discuss in the notes) asymptotically stable, then one also has to consider the nonlinear terms.*

8.4. Steady state response and center manifold

We first review the linear case. Consider a controllable and observable SISO linear system:

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx \end{aligned} \tag{8.5}$$

where $x \in R^n$. Suppose u is generated by the following exogenous system:

$$\begin{aligned} \dot{w} &= \Gamma w \\ u &= qw \end{aligned} \tag{8.6}$$

where $w \in R^m$ and $\sigma(\Gamma) \in \bar{C^+}$.

Then we already know from Chapter 6 that:

PROPOSITION 8.10. *Suppose A is a stable matrix, then all trajectories of $(x(t), w(t))$ tend asymptotically to the invariant subspace $S := \{(x, w) : x = \Pi w\}$, where Π is the solution of*

$$A\Pi - \Pi\Gamma = -bq.$$

On the invariant subspace, we have

$$y = c\Pi w.$$

Naturally $y^* := c\Pi w$ is the steady state response.

Now consider a nonlinear system in general

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned} \tag{8.7}$$

where x is defined in a neighborhood of the origin $N(0)$ of R^n, $u \in R^m$ and $f(0, 0) = 0$.

Let $u^*(t)$ be given and suppose there exists $x^* \in N(0)$ such that

$$\lim_{t\to\infty} \|x(t, x_0, u^*(\cdot)) - x(t, x^*, u^*(\cdot))\| = 0$$

$\forall\ x_0 \in N'(x^*)$ (some neighborhood of x^*), then

$$x_{ss}(t) = x(t, x^*, u^*(\cdot))$$

is called the steady state response to $u^*(\cdot)$.

We assume that $u^*(t)$ is generated by a dynamical system

$$\begin{aligned} \dot{w} &= s(w) \\ u &= p(w) \end{aligned}$$

and $w = 0$ is Liapunov stable, and

$$\frac{\partial s}{\partial w}|_{w=0}$$

has all eigenvalues on the imaginary axis.

Such a system is called an exogenous system or an exo-system.

EXAMPLE 8.5. *Consider*

$$\begin{aligned}\dot{x}_1 &= -x_1 + u\\ \dot{x}_2 &= -x_2 + x_1 u\end{aligned}$$

with desired u^* *to be*

$$u^*(t) = Acos(at) + Bsin(at)$$

Question: does $x_{ss}(t)$ *exist?*
u^* *can be generated by*

$$\begin{aligned}\dot{w}_1 &= aw_2\\ \dot{w}_2 &= aw_1\\ u^* &= w_1.\end{aligned}$$

Now consider the augmented system:

$$\begin{aligned}\dot{x}_1 &= -x_1 + w_1\\ \dot{x}_2 &= -x_2 + x_1 w_1\\ \dot{w}_1 &= aw_2\\ \dot{w}_2 &= aw_1\end{aligned}$$

For this system every solution tends to the center manifold:

$$\begin{aligned}x_1 = \pi_1(w) &= \frac{1}{1+a^2}(w_1 - aw_2)\\ x_2 = \pi_2(w) &= \frac{1}{1+5a^2+4a^4}((1+a^2)w_1^2 - 3aw_1w_2 + 3a^2w_2^2)\end{aligned}$$

So $x_{ss}(t) = (\pi_1(w), \pi_2(w))$*, with* $w_1(0) = A,\ w_2(0) = aB$*.*

PROPOSITION 8.11. *Suppose the system* (8.7) *is locally exponentially stable. Then there exists a mapping* $x = \pi(w)$ *defined in* $W(0)$ *with* $\pi(0) = 0$*, such that*

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), p(w))$$

for all $w \in W(0)$*. Moreover, the input*

$$u^*(t) = p(w(t))$$

with sufficiently small $w(0)$ *produces a steady state response*

$$x_{ss}(t) = \pi(w(t)).$$

We will not give a proof for this result, since it can be derived directly from the results in the next section.

8.5. Center Manifold Theory

Consider

$$\dot{x} = f(x) \tag{8.8}$$

where f is C^2 in $N(0) \in R^n$ and $f(0) = 0$.

Let

$$L = \frac{\partial f}{\partial x}|_{x=0},$$

then,

i. $x = 0$ is asymptotically stable if $\sigma(L) \in C^-$.

ii. $x = 0$ is unstable if at least one eigenvalue of L is in the right-half complex plane.

What happens to the critical case? i.e. the case where L has no eigenvalues in the right-half plane but has some eigenvalues on the imaginary axis.

Let us rewrite the above system (possibly after a linear coordinate change) as:

$$\begin{aligned} \dot{z} &= Az + f(z, y), \quad z \in \Re^p \\ \dot{y} &= By + g(z, y), \quad y \in \Re^m \end{aligned} \tag{8.9}$$

where A, B are constant matrices and $\sigma(A) \subset C^0$, $\sigma(B) \subset C^-$, i.e. A has its eigenvalues on the imaginary axis and B has its eigenvalues in the left half plane, $f(0,0) = 0$, $f'(0,0) = 0$, $g(0,0) = 0$, $g'(0,0) = 0$.
(f' denotes the Jacobian of f).

Definition 8.7. *Consider*

$$\dot{x} = f(x).$$

A set M is said to be an invariant *set of the system if for all initial conditions $x_0 \in M$, the trajectory*

$$x(x_0, t) \in M \qquad \forall t \geq 0.$$

Theorem 8.12. *There exists an invariant set defined by y=h(z), $\|z\| < \delta$ $h \in C^2$, and h(0)=0, h'(0)=0. This invariant set is called a* center manifold. *On the center manifold the dynamics of the system is governed by*

$$\dot{w} = Aw + f(w, h(w)) \tag{8.10}$$

Lemma 8.13. *Suppose $y = h(z)$ is a center manifold for* (8.9). *Then there exists a neighborhood N of 0, and $M > 0$, $k > 0$, such that $|y(t) - h(z(t))| \leq Me^{-kt}|y(0) - h(z(0))|$ for all $t \geq 0$, as long as $(y(t), z(t)) \in N$.*

Theorem 8.14. *The equilibrium (z,y)=(0,0) of* (8.9) *is unstable, stable or asymptotically stable if and only if w=0 of* (8.10) *is respectively unstable, stable or asymptotically stable.*

The proofs of the above three results are beyond the scope of the notes, but interested readers can find them in [**3**].

Now let $\Phi : \Re^p \to \Re^m$ be a C^1 mapping and define

$$[M\Phi](z) \stackrel{\triangle}{=} \Phi'(z)(Az + f(z, \Phi(z))) - B\Phi(z) - g(z, \Phi(z))$$

If Φ is a center manifold, then $[M\Phi](z) = 0$.

THEOREM 8.15. *If* $[M\Phi](z) = \mathcal{O}(|z|^q)$ *where* $q > 1$ *as* $z \to 0$*, then as* $z \to 0$

$$h(z) = \Phi(z) + \mathcal{O}(|z|^q)$$

EXAMPLE 8.6. *Determine if the equilibrium of the following system is asymptotically stable*

$$\begin{aligned} \dot{x}_1 &= x_1 x_2^3 \\ \dot{x}_2 &= -x_2 - x_1^2 \end{aligned}$$

We first try $x_2 = -x_1^2$ *as the approximation of a center manifold. Then*

$$[M\Phi](x_1) = -2x_1(-x_1 x_1^6) - 0 = -2x_1^8$$

So $h(x_1) = -x_1^2 + \mathcal{O}(x_1^8)$ *on the center manifold*

$$\dot{w} = wh^3(w) = -w^7 + \mathcal{O}(w^{13})$$

$w = 0$ *is asymptotically stable. Therefore,* $(x_1, x_2) = (0, 0)$ *is asymptotically stable.*

EXAMPLE 8.7. *Consider*

$$\begin{aligned} \dot{x}_1 &= x_1 x_2^3 \\ \dot{x}_2 &= -x_2 - x_1^2 \\ \dot{x}_3 &= -x_3^3 + x_1 r(x_1, x_2, x_3) \end{aligned}$$

The center manifold is same as in the previous example and the flow on the center manifold is governed by

$$\begin{aligned} \dot{w}_1 &= -w_1^7 + \mathcal{O}(w_1^{13}) \\ \dot{w}_2 &= -w_2^3 + w_1 r(w_1, h(w_1), w_2) \end{aligned}$$

The following lemma is useful when one needs to decide stability on the center manifold.

LEMMA 8.16. *Consider*

$$\dot{z} = f(z, y) \tag{8.11}$$

$$\dot{y} = g(y) \tag{8.12}$$

Suppose $y = 0$ *of* $\dot{y} = g(y)$ *and* $z = 0$ *of* $\dot{z} = f(z, 0)$ *are asymptotically stable, then* $(z, y) = (0, 0)$ *of* (8.11) *is asymptotically stable.*

8.6. Zero dynamics and its applications

Consider

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

where $y \in R$, $u \in R$, $x \in R^n$.

The system is said to have relative degree r at a point x^0 if

i. $L_g L_f^k h(x) = 0$, $\forall x \in N(x^0)$ and $k < r-1$,

ii. $L_g L_f^{r-1} h(x^0) \neq 0$.

Lemma 8.17. *The row vectors*

$$dh(x^0), dL_f h(x^0), \ldots, dL_f^{r-1} h(x^0)$$

are linearly independent if the system has relative degree r.

Normal form:

Suppose the system has relative degree r ($r < n$) at x^0, then at $N(x^0)$ it can be transformed into the following form by a nonlinear coordinate change:

$$\begin{aligned} \dot{z} &= f_0(z, \xi) \qquad z \in R^{n-r} \cap N(x^0) \\ \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_{r-1} &= \xi_r \\ \dot{\xi}_r &= f_1(z, \xi) + g_1(z, \xi)u \end{aligned}$$

where $\xi_i = L_f^{i-1} h(x)$.

How to find the coordinate changes of $z(x)$? Since $G = span\ \{g(x)\}$ is involutive, we have

$$G^{\perp} = span\ \{dh, \cdots, dL_f^{r-2} h, dz_1(x), \cdots, dz_{n-r}(x)\}.$$

One might need to make several tries before those closed one-forms can be obtained.

The name "zero dynamics" is due to its relation to transmission zeros of a linear system. Naturally, for nonlinear systems transmission zeros do not make any sense.

Now let us consider the following problem:

Suppose the system has relative degree r, find a control $u(t)$ and/or initial conditions such that $y(t) = 0$ $\forall t \geq 0$.

$h(x) = 0$ $\forall t \geq 0$ implies

$$\dot{h}(x) = L_f h(x) + L_g h(x) u = 0$$

If $r = 1$ or $L_g h(x^0) \neq 0$, then

$$u = -\frac{L_f h(x)}{L_g h(x)}$$

Otherwise the initial conditions must be in

$$Z^* = \{x \in N(x^0) : h(x) = L_f h(x) = \cdots = L_f^{r-1} h(x) = 0\}$$

and

$$u = -\frac{L_f^r H(x)}{L_g L_f^{r-1} h(x)}$$

DEFINITION 8.8 (Zero dynamics). *The dynamics of the system restricted to Z^* is called the* zero dynamics*:*

$$\dot{z} = f_0(z, 0).$$

8.6.1. Local feedback stabilization. Consider

$$\dot{x} = f(x) + g(x)u \tag{8.13}$$

$$y = h(x) \tag{8.14}$$

where $x \in N(0) \in R^n,\ f(0) = 0,\ f \in C^1,\ g \in C^1$ and $f(0) = 0,\ h(0) = 0$.

REMARK 8.3. *although in this section our focus is on SISO systems, all the results discussed in the rest of this section also apply to MIMO systems.*

The system (8.13) is said to be locally stable if the origin (0) of the system is asymptotically stable and the domain of the attraction (the set of initial conditions from which the solution tends to the origin) is not necessarily the whole R^n space.

Let

$$A = \frac{\partial f(0)}{\partial x}, \quad b = g(0)$$

Fact: if the pair(A, b) is stabilizable, then (8.13) is locally stabilizable.

For linear systems we know that as long as the system is controllable, then it is stabilizable. For nonlinear systems the situation is much more complex. Nonlinear controllability does not necessarily imply stabilizability by differentiable feedback controls.

PROPOSITION 8.18. *A necessary condition for* (8.13) *to be stabilizable by a C^1 feedback control is*

a. The linearization pair (A, b) does not have uncontrollable modes which are associated with the unstable eigenvalues.

b. The map $(x, u) \to f(x) + g(x)u$ is onto a neighborhood of 0.

The above result is sometimes called the Brockett theorem [**2**]. The theoretical foundation for this was also given by a Russian mathematician Krasnoselskii [**12**].

PROPOSITION 8.19. *Consider a system in the normal form in $N(0)$ of R^n:*

$$\begin{aligned}\dot{z} &= f_0(z,\xi)\\ \dot{\xi}_1 &= \xi_2\\ &\vdots\\ \dot{\xi}_{r-1} &= \xi_r\\ \dot{\xi}_r &= f_1(z,\xi)+g_1(z,\xi)u\end{aligned}$$

If the zero dynamics of the system is locally asymptotically stable, then the system is locally stabilizable. A stabilizing control is

$$u=\frac{1}{g_1(z,\xi)}(-f_1(z,\xi)-a_r\xi_1+\cdots-a_1\xi_r),$$

where a_i, $i=1,\dots,r$ are chosen so that the polynomial

$$s^r+a_1s^{r-1}+\cdots+a_r$$

becomes Hurwitz polynomial (i.e., all the roots are in the open left half-plane.)

8.6.2. Limitation of high gain control. For a linear control system, if it is minimum phase and has relative degree one, it is well known that a high gain output control can be used to stabilize the system. Is this still true for nonlinear systems?

Let us consider the following example:

$$\begin{aligned}\dot{z} &= y-z^3\\ \dot{y} &= z+u\end{aligned}$$

The zero dynamics is

$$\dot{z}=-z^3.$$

Now use high gain control $u=-ky$ where $k>0$. Then

$$\begin{aligned}\dot{z} &= y-z^3\\ \dot{y} &= z-ky\end{aligned}$$

The closed-loop system is not stable for any k!

We observe that for this system the zero dynamics is only critically (namely not exponentially) asymptotically stable. In fact, this is precisely the reason why the high gain control does not work.

PROPOSITION 8.20. *Consider system* (8.13)*. If it has relative degree one at $x=0$ and its zero dynamics is exponentially stable, then the high gain control $u=-ky$, where $k>0$ if $L_gh(0)>0$ and $k<0$ if $L_gh(0)<0$, locally stabilizes the system when $|k|$ is sufficiently large.*

8.7. Disturbance decoupling problem (DDP)

Consider a SISO system with disturbance:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + p(x)w \\ y &= h(x) \end{aligned}$$

where w is the disturbance.

DEFINITION 8.9 (DDP). *Find an input feedback transformation*

$$u = \alpha(x) + \beta(x)v$$

such that the output y is completely decoupled form w.

PROPOSITION 8.21. *Suppose the system has relative degree r at x^0. Then the DDP is locally solvable if and only if $L_p L_f^i h(x) = 0, \ \forall i \leq r-1, \ \forall x \in N(x^0)$.*

Another way to say it:

PROPOSITION 8.22. *Let $\Omega = span\{dh, \cdots, dL_f^{r-1}h\}$, then DDP is locally solvable iff*

$$p(x) \in \Omega^{\perp}(x), \quad \forall x \in N(x^0).$$

8.8. Output regulation

Consider

$$\dot{x} = f(x) + g(x)u + p(x)w \tag{8.15}$$
$$\dot{w} = s(w) \tag{8.16}$$
$$e = h(x, w) \tag{8.17}$$

where the first equation is the plant with $f(0) = 0$, the second equation is an exosystem as we defined before and e is the tracking error. Here w represents both the signals to be tracked and disturbances to be rejected.

Full information output regulation problem

Find, if possible, $u = \alpha(x, w)$, such that

1. $x = 0$ of

$$\dot{x} = f(x) + g(x)\alpha(x, 0)$$

is exponentially stable;

2. the solution to

$$\begin{aligned} \dot{x} &= f(x, w, \alpha(x, w)) \\ \dot{w} &= s(w) \end{aligned}$$

satisfies

$$\lim_{t \to \infty} e(x(t), w(t)) = 0$$

for all initial data in some neighborhood of the origin.

Let us first recall the linear case that was studied earlier:

$$\begin{aligned} \dot{x} &= Ax + Bu + Pw \\ \dot{w} &= Sw \\ e &= Cx + Qw \end{aligned}$$

PROPOSITION 8.23. *Suppose the pair (A, B) is stabilizable and no eigenvalue of S is on the open left half plane, then the full information output regulation problem is solvable if and only if there exist matrices Π and Γ which solve the linear matrix equation*

$$\begin{aligned} A\Pi + B\Gamma + P &= \Pi S \\ C\Pi + Q &= 0. \end{aligned}$$

The feedback control then is

$$u = K(x - \Pi w) + \Gamma w$$

where $A + BK$ is Hurwitz.

Now consider (8.15). Suppose:

H1: $w = 0$ is a stable equilibrium of the exosystem and

$$\frac{\partial s}{\partial w}|_{w=0}$$

has all eigenvalues on the imaginary axis.

H2: The pair $f(x),\ g(x)$ has a stabilizable linear approximation at $x = 0$.

THEOREM 8.24. *Suppose $H1$ and $H2$ are satisfied. The full information output regulation problem is solvable if and only if there exist $\pi(w),\ c(w)$ with $\pi(0) = 0,\ c(0) = 0$, both defined in some neighborhood of the origin, satisfying the equations*

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(\pi(w)) + g(\pi(w))c(w) + p(\pi(w))w \\ h(\pi(w), w) &= 0 \end{aligned}$$

The feedback control can be designed as

$$\alpha(x, w) = K(x - \pi(w)) + c(w)$$

where K stabilizes the linearization of $\dot{x} = f(x) + g(x)u$ in (8.15).

8.9. Exact linearization via feedback

Consider

$$\dot{x} = f(x) + g(x)u, \quad x \in N(x^0) \in R^n$$

DEFINITION 8.10 (Exact linearization problem). *Find $u = \alpha(x) + \beta(x)v$ and a coordinate change $z = \phi(x)$ such that*

$$\dot{x} = f(x) + \alpha(x)g(x) + \beta(x)g(x)v$$

can be transformed into

$$\dot{z} = Az + bv$$

where (A, b) **is controllable.**

PROPOSITION 8.25. *The exact linearization problem is solvable at* x^0 *iff there exists a real-valued function* $\lambda(x)$ *defined on* $N(x^0)$*, s.t.*

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= \lambda(x) \end{aligned}$$

has relative degree n *at* x^0.

PROPOSITION 8.26. *The exact linearization problem is solvable at* x^0 *iff*
i. the matrix $[g(x^0)\ ad_f g(x^0)\ \cdots\ ad_f^{n-1} g(x^0)]$ *has rank* n,
ii. the distribution $D = span\{g, ad_f g, \cdots, ad_f^{n-2} g\}$ *is involutive in* $N(x^0)$.
where $ad_f^0 g := g$, $ad_f^{k+1} g = [f, ad_f^k g]$.

Proposition 8.26 guarantees that we can find such a $\lambda(x)$ as the "dummy" output for the system so that the system can be transformed into the normal form with relative degree n, thus linearizable. Now the question is *How to find* $\lambda(x)$*?*

We begin by computing $D^{\perp}$. Since D has rank $n-1$, we have

$$D^{\perp} = span\ \omega(x),$$

where $\omega(x)$ is a one-form that does not vanish. If ω is exact, then we can integrate ω to get λ. Otherwise let $\omega_1(x) = c(x)\omega(x)$, where $c(x)$ is a non-zero scalar function. Since $D(x)$ is involutive, we are guaranteed that there exists such a $c(x)$ that $\omega_1(x)$ is exact.