Mathematical Systems Theory: Advanced Course
Exercise Session 6

1 How to check stability in critical cases?

Consider a nonlinear system

$$\dot{x} = f(x).$$

Now suppose that the matrix

$$L := \left.\frac{\partial f}{\partial x}\right|_{x=0}$$

has no eigenvalues in the open right half-plane but has some eigenvalues on the imaginary axis. Such cases are called *critical cases*. To check the stability in such cases, one can use the center manifold theory.

The procedure to check the stability is as follows.

Step 1. From $\dot{x} = f(x)$, obtain

$$\dot{x} = Lx + p(x), \tag{1}$$

where p includes higher order terms than order one.

Step 2. If necessary, do a coordinate change to transform (1) into

$$\begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A & \\ & B \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix} + \begin{bmatrix} f(z,y) \\ g(z,y) \end{bmatrix},$$

where A and B have eigenvalues only on the imaginary axis and only in the open left half-plane, respectively.

Step 3. First, try to solve $\dot{y} = 0$, i.e.,

$$By + g(z,y) = 0,$$

with respect to y. If it is difficult to solve, we solve instead $By + g(z,0) = 0$, i.e.,

$$y = -B^{-1}g(z,0).$$

Set $\phi(z) := y$. Using the obtained ϕ, define

$$M\phi(z) := \frac{\partial \phi}{\partial z}(Az + f(z,\phi(z))) - B\phi(z) - g(z,\phi(z))$$

Step 4. The center manifold is approximated as

$$h(z) = \phi(z) + \mathcal{O}(M\phi(z)).$$

Step 5. Check the stability of

$$\dot{w} = Aw + f(w, h(w)).$$

Example 1

Check the stability of the system

$$\begin{cases} \dot{x}_1 &= x_1^4 + x_1 x_2 \\ \dot{x}_2 &= -2x_2 - x_1^2 + x_1 x_2^2 \end{cases}$$

This system can be written as

$$\begin{cases} \dot{x}_1 &= \underbrace{0}_{A} x_1 + \underbrace{x_1^4 + x_1 x_2}_{f(x_1,x_2)} \\ \dot{x}_2 &= \underbrace{-2}_{B} x_2 + \underbrace{(-x_1^2) + x_1 x_2^2}_{g(x_1,x_2)}. \end{cases}$$

Since it is difficult to solve $-2x_2 - x_1^2 + x_1 x_2^2 = 0$ with respect to x_2, we set

$$x_2 = -B^{-1} g(x_1, 0) = -\frac{1}{2} x_1^2 =: \phi(x_1).$$

Hence,

$$\begin{aligned} M\phi(x_1) &:= \frac{\partial \phi}{\partial x_1}(Ax_1 + f(x_1, \phi(x_1))) - B\phi(x_1) - g(x_1, \phi(x_1)) \\ &= -x_1 \left(x_1^4 - \frac{1}{2} x_1^3 \right) - \frac{1}{4} x_1^5 \\ &= \mathcal{O}(x_1^4). \end{aligned}$$

So, the center manifold is approximated as

$$h(x_1) = -\frac{1}{2} x_1^2 + \mathcal{O}(x_1^4).$$

Let us check the stability of

$$\begin{aligned} &\dot{w} = \underbrace{0}_{A} w + \underbrace{w^4 + w \left(-\frac{1}{2} w^2 + \mathcal{O}(w^4) \right)}_{f(w, h(w))} \\ &\Rightarrow \dot{w} = -\frac{1}{2} w^3 + \mathcal{O}(w^4). \end{aligned}$$

Since $w = 0$ is asymptotically stable for this system, $(x_1, x_2) = (0, 0)$ is also asymptotically stable for the original system.

Example 2

Consider the control system

$$\begin{cases} \dot{x}_1 &= x_2 x_3 \\ \dot{x}_2 &= u_1 \\ \dot{x}_3 &= -x_1 x_2 + u_2. \end{cases}$$

This is the model of spacecraft with some constants (see the note for Exercise Session 5). If we use control

$$u_1 = -x_2 + x_1^2, \quad u_2 = -x_3 - x_1^3,$$

the closed-loop system becomes

$$\begin{cases} \dot{x}_1 &= x_2 x_3 \\ \dot{x}_2 &= -x_2 + x_1^2 \\ \dot{x}_3 &= -x_1 x_2 - x_3 - x_1^3. \end{cases}$$

We will check the stability of this closed-loop system.

We can write the system as

$$\begin{cases} \dot{x}_1 &= A x_1 + f(x_1, [x_2, x_3]) \\ \begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= B \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + g(x_1, [x_2, x_3]) \end{cases}$$

where

$$A = 0,\ B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ f(x_1, [x_2, x_3]) = x_2 x_3,\ g(x_1, [x_2, x_3]) = \begin{bmatrix} x_1^2 \\ -x_1 x_2 - x_1^3 \end{bmatrix}.$$

First, solve

$$0 = -x_2 + x_1^2 \Rightarrow x_2 = x_1^2 =: \phi_1(x_1)$$

$$0 = -x_1 x_2 - x_3 - x_1^3 \Rightarrow x_3 = -2x_1^3 =: \phi_2(x_1)$$

Define $\phi(x_1) := \begin{bmatrix} \phi_1(x_1) \\ \phi_2(x_1) \end{bmatrix}$. Then,

$$\begin{aligned} M\phi(x_1) &:= \frac{\partial \phi}{\partial x_1}(A x_1 + f(x_1, \phi(x_1))) - B\phi(x_1) - g(x_1, \phi(x_1)) \\ &= \begin{bmatrix} 2x_1 \\ -6x_1^2 \end{bmatrix} x_1^2(-2x_1^3) = \begin{bmatrix} \mathcal{O}(x_1^6) \\ \mathcal{O}(x_1^7) \end{bmatrix}. \end{aligned}$$

So, the center manifold is approximated as

$$h(x_1) = \begin{bmatrix} x_1^2 \\ -2x_1^3 \end{bmatrix} + \begin{bmatrix} \mathcal{O}(x_1^6) \\ \mathcal{O}(x_1^7) \end{bmatrix}.$$

Let us check the stability of the system

$$\begin{aligned} &\dot{w} = (w^2 + \mathcal{O}(w^6))(-2w^3 + \mathcal{O}(w^7)) \\ &\Rightarrow \dot{w} = -2w^5 + \mathcal{O}(w^9). \end{aligned}$$

$w = 0$ of this system is asymptotically stable, and so is $x = 0$ of the original system.

2 Normal form in SISO nonlinear systems

Consider a SISO nonlinear system

$$\begin{cases} \dot{x} &= f(x) + g(x)u \\ y &= h(x). \end{cases}$$

The system has *relative degree at a point* x_0 if

$$\begin{aligned} &L_g L_f^k h(x) = 0, \forall x \in \mathcal{N}(x_0),\ k = 0, 1, \ldots, r-2, \\ &L_g L_f^{r-1} h(x_0) \neq 0. \end{aligned}$$

If the system has relative degree at x_0, then in $\mathcal{N}(x_0)$, we can transform the system into a normal form:

$$\begin{cases} \dot{z} &= f_0(z, \xi), \\ \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_{r-1} &= \xi_r \\ \dot{\xi}_r &= f_1(z, \xi) + g_1(z, \xi)u. \end{cases}$$

The zero dynamics is

$$\dot{z} = f_0(z, 0).$$

To obtain a normal form, we take new states as

$$\xi_1 := h(x), \xi_2 := L_f h(x), \cdots, \xi_r := L_f^{r-1} h(x).$$

As for the z part, first define

$$\mathcal{D} := \operatorname{span}\{g\}.$$

Then, compute

$$\mathcal{D}^{\perp} := \{w_i(x) : \ i = 1, \ldots, n-1, \ w_i(x)g = 0\}.$$

For each row vector $w_i(x) =: \begin{bmatrix} w_1^i(x) & \cdots & w_n^i(x) \end{bmatrix}$, if the following holds:

$$\frac{\partial w_j^i}{\partial x_k} = \frac{\partial w_k^i}{\partial x_j}, \ \forall j, k,$$

then you can find z_i satisfying

$$dz_i = w_i.$$

Choose such z_i that are linearly independent of ξ part that has already been chosen.

Otherwise, you have to change the basis of $\mathcal{D}^{\perp}$. (But how to find such basis is not required in this course.)

Example

Consider the system

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin x_1 + u \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \sin 2x_1 + (\cos x_1)u \\ y &= x_1, \end{cases}$$

or equivalently,

$$\begin{cases} \dot{x} &= \underbrace{\begin{bmatrix} x_2 \\ \sin x_1 \\ x_4 \\ \sin 2x_1 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \cos x_1 \end{bmatrix}}_{g(x)} u \\ y &= \underbrace{x_1}_{h(x)} \end{cases}$$

First, let us check if the system has relative degree at $x = 0$.

$$\begin{aligned} L_g h(x) &= \frac{\partial h}{\partial x} g = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} g = 0 \\ L_g L_f h(x) &= L_g \left(\frac{\partial h}{\partial x} f \right) = L_g(x_2) = \frac{\partial x_2}{\partial x} g = 1 \neq 0. \end{aligned}$$

Hence, relative degree is two.

Next, we transform the system into a normal form. We take new states as

$$\xi_1 := h(x) = x_1, \ \xi_2 := L_f h(x) = x_2.$$

We have to take another two states z_1 and z_2 (z part). To this end, we first find

$$\mathcal{D}^{\perp} := (\mathrm{span}\,\{g\})^{\perp} = \mathrm{span}\left\{e_1^T, e_3^T, \begin{bmatrix} * & \cos x_1 & * & -1 \end{bmatrix}\right\}.$$

We obtain one state z_1 from the following observation:

$$\begin{array}{l} dz = e_1^T \Rightarrow z_1 = x_1 \text{ (already chosen as } \xi_1\text{. Ignore!)} \\ dz = e_3^T \Rightarrow z_1 = x_3. \end{array}$$

To ensure the existence of z_2 with $dz_2 = \begin{bmatrix} * & \cos x_1 & * & -1 \end{bmatrix}$, we verify

$$\frac{\partial \cos x_1}{\partial x_4} = \frac{\partial(-1)}{\partial x_2} (= 0).$$

So we can solve

$$dz_2 = \begin{bmatrix} * & \cos x_1 & * & -1 \end{bmatrix}.$$

or equivalently,

$$\begin{cases} \dfrac{\partial z_2}{\partial x_2} & = & \cos x_1 \\ \dfrac{\partial z_2}{\partial x_4} & = & -1 \end{cases}$$

One solution is

$$z_2 = (\cos x_1) x_2 - x_4.$$

Since $\xi_1 := x_1$, $\xi_2 := x_2$ and $z_1 := x_3$ do not include x_4, this z_2 satisfies the second condition above.

Therefore,

$$\begin{aligned} \dot{z}_1 &= \dot{x}_3 = x_4 = (\cos x_1) x_2 - z_2 = (\cos \xi_1) \xi_2 - z_2 \\ \dot{z}_2 &= (-\sin x_1) \dot{x}_1 x_2 + (\cos x_1) \dot{x}_2 - \dot{x}_4 \\ &= -(\sin x_1) x_2^2 + (\cos x_1)(\sin x_1 + u) - (\sin 2x_1 + (\cos x_1) u) \\ &= -(\sin \xi_1) \xi_2^2 - \frac{1}{2} \sin 2\xi_1 \\ \dot{\xi}_1 &= \dot{x}_1 = x_2 = \xi_2 \\ \dot{\xi}_2 &= \dot{x}_2 = \sin x_1 + u = \sin \xi_1 + u \\ y &= \xi_1. \end{aligned}$$

The zero dynamics is obtained by setting $\xi = 0$:

$$\begin{aligned}\dot{z}_1 &= -z_2 \\ \dot{z}_2 &= 0.\end{aligned}$$