Mathematical Systems Theory: Advanced Course
Exercise Session 7

1 Local feedback stabilization

Consider a nonlinear control system

$$\dot{x} = f(x) + g(x)u.$$

To check the local stabilizability of this system, follow the procedure below.

1. First, you should **always** check if the linearized system with

$$A := \frac{\partial f}{\partial x}(0),\ b = g(0)$$

   is controllable (or stabilizable). If it is, then the nonlinear system is locally stabilizable.

2. If Step 1 fails, then use Proposition 8.19 (page 74) in case the system can be transformed into a normal form:

$$\begin{aligned}
\dot{z} &= f_0(z, \xi) \\
\dot{\xi}_1 &= \xi_2 \\
&\vdots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= f_1(z, \xi) + g_1(z, \xi)u \\
y &= \xi_1.
\end{aligned}$$

   If the zero dynamics of the system is locally asymptotically stable, then the stabilizing control is

$$u = \frac{1}{g_1(z, \xi)}(-f_1(z, \xi) - a_r\xi_1 + \cdots - a_1\xi_r),$$

   where a_i, $i = 1, \ldots, r$ are chosen so that the polynomial

$$s^r + a_1 s^{r-1} + \cdots + a_r$$

   becomes Hurwitz polynomial (i.e., all the roots are in the open left half-plane.)

2 Output regulation

Consider

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)u + p(x)w \\
\dot{w} &= s(w) \\
e &= h(x, w)
\end{aligned}
$$

where the first equation is the plant with $f(0) = 0$, the second equation is an exosystem as we defined before and e is the tracking error. Here w represents both the signals to be tracked and disturbances to be rejected.

In the course we only consider the following full information output regulation problem:

Find, if possible, $u = \alpha(x, w)$, such that

1. $x = 0$ of

$$\dot{x} = f(x) + g(x)\alpha(x, 0)$$

is exponentially stable;

2. the solution to

$$
\begin{aligned}
\dot{x} &= f(x, w, \alpha(x, w)) \\
\dot{w} &= s(w)
\end{aligned}
$$

satisfies

$$\lim_{t \to \infty} e(x(t), w(t)) = 0$$

for all initial data in some neighborhood of the origin.

Solvability condition (Theorem 8.24)

Suppose 1. $w = 0$ is a stable equilibrium of the exosystem and

$$\frac{\partial s}{\partial w}|_{w=0}$$

has all eigenvalues on the imaginary axis, and 2. the pair $f(x),\ g(x)$ has a stabilizable linear approximation at $x = 0$. Then the full information output regulation problem is solvable if and only if there exist $\pi(w),\ c(w)$ with $\pi(0) = 0,\ c(0) = 0$, both defined in some neighborhood of the origin, satisfying the equations

$$
\begin{aligned}
\frac{\partial \pi}{\partial w} s(w) &= f(\pi(w)) + g(\pi(w))c(w) + p(\pi(w))w \\
h(\pi(w), w) &= 0
\end{aligned}
$$

The feedback control can be designed as

$$\alpha(x, w) = K(x - \pi(w)) + c(w)$$

where K stabilizes the linearization of $\dot{x} = f(x) + g(x)u$.

Example

Consider the system

$$\begin{aligned}
\dot{x}_1 &= -x_1 + x_2 w_1 \\
\dot{x}_2 &= -x_1 - x_2 + u \\
\dot{w}_1 &= w_2 \\
\dot{w}_2 &= -w_1 \\
e &= x_2 - \frac{1}{2}w_1 - \frac{1}{2}w_2.
\end{aligned}$$

In order to solve the output regulation problem we need to find $\pi(w)$ and $c(w)$.

Let $x_1 = \pi_1(w), x_2 = \pi_2(w)$. Then from $e = 0$ we have

$$\pi_2(w) = \frac{1}{2}w_1 + \frac{1}{2}w_2,$$

which implies (by taking derivative on both sides of $x_2 = \pi_2(w)$),

$$-\pi_1(w) - \pi_2(w) + c(w) = \frac{1}{2}w_2 - \frac{1}{2}w_1.$$

Thus, $c(w) = \frac{1}{2}w_2 - \frac{1}{2}w_1 + \pi_1 + \pi_2$.

Now we need to decide π_1 (from $\dot{x}_1 = -x_1 + x_2 w_1$):

$$\frac{\partial \pi_1(w)}{\partial w} \cdot (w_2 \;\; -w_1)^T = -\pi_1 + w_1 \pi_2.$$

Since π_2 is linear in w, one can easily determine that $\pi_2 = c_1 w_1 + c_2 w_2 + c_3 w_1^2 + c_4 w_2^2 + c_5 w_1 w_2$. By plugging in this into the above equation, we have

$$\pi_2 = \frac{1}{10}(2w_1^2 + 3w_2^2 - 3w_1 w_2).$$

Since the system with zero input and disturbance is already asymptotically stable, we can choose $K = 0$, thus

$$\alpha = c(w)$$

solves the output regulation problem.

3 Exact linearization

Consider a nonlinear control system

$$\dot{x} = f(x) + g(x)u, \ x \in \mathcal{N}(x^0) \subset R^n$$

We want to find

- a feedback $u = \alpha(x) + \beta(x)v$, and
- a coordinate change $z = \phi(x)$,

so that the resulting system becomes a linear system:

$$\dot{z} = Az + bv,$$

where (A, b) is controllable.

Solvability condition (Proposition 8.26)

The exact linearization problem is solvable at x^0 if and only if

1. rank $\left[\begin{array}{cccc} g(x^0) & ad_f g(x^0) & \cdots & ad_f^{n-1} g(x^0) \end{array} \right] = n$
2. The distribution $\mathcal{D}(x) :=$ span $\left\{ g(x), ad_f g(x), \cdots, ad_f^{n-2} g(x) \right\}$ is involutive in $N(x^0)$.

Here,

$$ad_f^0 g := g, \ ad_f^1 g := [f, g], \ ad_f^{k+1} g := \left[f, ad_f^k g \right],$$

and $\mathcal{D}$ is *involutive* if for any $k_1, k_2 \in \mathcal{D}$,

$$[k_1, k_2] \in \mathcal{D}.$$

How to solve the exact linearization

To solve the exact linearization, we use proposition 8.23 in the lecture notes, we want to find $\lambda(x)$ such that

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= \lambda(x) \end{aligned}$$

has relative degree n at x_0.

How to obtain λ is explained through the example.
When we have λ, we can transform the system into a normal form:

$$\begin{cases} \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_{n-1} &= \xi_n \\ \dot{\xi}_n &= L_f^n\lambda + L_g L_f^{n-1}\lambda u \\ y &= \xi_1. \end{cases}$$

by doing the following coordinate change

$$\begin{aligned} \xi_1 &:= \lambda(x) \\ \xi_2 &:= L_f\lambda(x) \\ &\vdots \\ \xi_n &:= L_f^{n-1}\lambda(x) \end{aligned}$$

We see now that the following feedback

$$u = -\frac{L_f^n\lambda}{L_g L_f^{n-1}\lambda} + v,$$

linearizes the system.

Example

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_3 \sin^2 x_1 + u \\ \dot{x}_2 &= 2x_3 \cos^2 x_1 - 2u \\ \dot{x}_3 &= 2\sin x_2, \end{aligned}$$

namely,

$$\dot{x} = \underbrace{\begin{bmatrix} x_3 \sin^2 x_1 \\ 2x_3 \cos^2 x_1 \\ 2\sin x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}}_{g} u$$

First, using Proposition 8.24, we check the solvability of the exact linearization at $x = 0$.

1. $ad_f g(0)$ and $ad_f^2 g(0)$ are computed as

$$\begin{aligned} ad_f g(0) &= [f,g]_{x=0} = \cdots = \begin{bmatrix} -x_3 \sin 2x_1 \\ 2x_3 \sin 2x_1 \\ 2\cos x_2 \end{bmatrix}_{x=0} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \\ ad_f^2 g(0) &= [f, ad_f g]_{x=0} = \cdots = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}. \end{aligned}$$

Hence,

$$\operatorname{rank} \begin{bmatrix} g(0) & ad_f g(0) & ad_f^2 g(0) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix} = 3.$$

2. Check if the distribution $\mathcal{D} := \text{span}\{g, ad_f g\}$ is involutive in $\mathcal{N}(0)$.

$$\begin{aligned} [g, ad_f g] &= \frac{\partial ad_f g}{\partial x} g - \frac{\partial g}{\partial x} ad_f g = \begin{bmatrix} -2x_3 \cos 2x_1 \\ 4x_3 \cos 2x_1 \\ 4 \sin x_2 \end{bmatrix}. \\ &= 2x_3(\tan x_2 \sin 2x_1 - \cos 2x_1) \underbrace{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}}_{g} + 2\tan x_2 \underbrace{\begin{bmatrix} -x_3 \sin 2x_1 \\ 2x_3 \sin 2x_1 \\ 2\cos x_2 \end{bmatrix}}_{ad_f g} \\ &\in \mathcal{D}. \end{aligned}$$

Hence $\mathcal{D}$ is involutive in $\mathcal{N}(0)$ $\rightarrow$ exact linearization solvable.

We want to find $\lambda(x)$ such that the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= \lambda(x) \end{aligned}$$

has relative degree three. Such λ is obtained by finding $\mathcal{D}^\perp$:

$$\mathcal{D}^\perp = \text{span}\{w\} = \text{span}\left\{ \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \right\}.$$

In this case, since w is a constant vector, there exists a λ satisfying

$$d\lambda = w.$$

Such λ can be easily found by inspection.

$$\lambda = 2x_1 + x_2.$$

With the obtained λ, the system has relative degree three. Hence, by doing a coordinate change as

$$\begin{aligned} \xi_1 &:= \lambda(x) = 2x_1 + x_2 \\ \xi_2 &:= L_f\lambda(x) = 2x_3 \\ \xi_3 &:= L_f^2\lambda(x) = 4\sin x_2, \end{aligned}$$

we can transform the system into a normal form:

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \dot{\xi}_3 = L_f^3\lambda + L_g L_f^2 \lambda u \\ y = \xi_1. \end{cases}$$

Thus, the exact linearization can be achieved by the feedback

$$u = -\frac{L_f^3\lambda}{L_g L_f^2\lambda} + v,$$

and the coordinate change above.

3.1 Multi-agent consensus

Consider N agents

$$\dot{x}_i = u_i, \quad i = 1, \cdots, N.$$

Suppose each agent uses the following neighborhood control:

$$u_i = \sum_{j \in N_i} (x_j - x_i),$$

where N_i indicates the neighbors of agent i.

We say the consensus is reached if as $t \to \infty$ we have

$$x_1(t) = x_2(t) = \cdots = x_N(t).$$

Solvability condition(Proposition 9.2)

The consensus problem is solved if the associated neighborhood graph is connected.

Example

We consider a three-agent system:

$$\dot{x}_i = u_i, \ i = 1, 2, 3.$$

Case 1: $N_1 = 2, N_2 = \{1, 3\}, N_3 = 2$. Then

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1 \\ \dot{x}_2 &= x_1 - x_2 + x_3 - x_2 \\ \dot{x}_3 &= x_2 - x_3. \end{aligned}$$

Let $\bar{x} = Px$, where

$$P = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

then

$$\bar{A} = PAP^{-1} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Clearly, A has one eigenvalue at zero and two eigenvalues at -2.

Case 2: $N_1 = \{2, 3\}, N_2 = \{1, 3\}, N_3 = \{1, 2\}$. Then

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1 + x_3 - x_1 \\ \dot{x}_2 &= x_1 - x_2 + x_3 - x_2 \\ \dot{x}_3 &= x_1 - x_3 + x_2 - x_3. \end{aligned}$$

Once again we let $\bar{x} = Px$, then

$$\bar{A} = PAP^{-1} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 1 & 3 & 0 \end{pmatrix}.$$

In this case A has one eigenvalue at zero and two eigenvalues at -3. This suggests that with more information available, the agents reach consensus faster.