Geometric Control Theory[1]

Lecture notes by

Xiaoming Hu and Anders Lindquist

in collaboration with

Jorge Mari and Janne Sand

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[1]This work is partially based on the earlier lecture notes by Lindquist, Mari and Sand.

Contents

Preface

This has been the set of lecture notes for the course "Matematisk System-teori, fortsättningskurs" given at the Royal Institute of Technology (KTH), Stockholm. The first version was written by Anders Lindquist and Janne Sand, and was later revised and extended by Jorge Mari. A major revision and addition was done by Xiaoming Hu in 2002.

I would like to express my gratitude to Dr Ryozo Nagamune and Docent Ulf Jönsson for their careful reading and constructive comments on the last revisions of the notes[2].

Some minor changes have been made since 2002. In 2006, some new material was added and the title was changed to "Geometric Control Theory".

Xiaoming Hu
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[2]This document has been generated with LaTeX2$_\epsilon$ and $\mathcal{AMS}$-LaTeX.

Notations and definitions

For easy reference we recall here some notations and definitions. Assume $\mathcal{X}$ and $\mathcal{Y}$ are finite dimensional vector spaces over the real field. Let n be the dimension of $\mathcal{X}$, which is then isomorphic to $\mathbb{R}^n$.

(1) *Image space.* Given a map $A : \mathcal{X} \to \mathcal{Y}$, the *image space* of A is defined as

$$\operatorname{Im} A := \{y \in \mathcal{Y} : y = Ax, \text{ for some } x \in \mathcal{X}\}.$$

It is a subspace of $\mathcal{Y}$. We shall often use the same symbol for the map A as for its matrix representation once bases in $\mathcal{X}$ and $\mathcal{Y}$ have been chosen.

(2) *Null space.* Given a map $A : \mathcal{X} \to \mathcal{Y}$, the *null space* , or *kernel*, of A is defined as

$$\ker A := \{x \in \mathcal{X} : Ax = 0\}.$$

It is a subspace of $\mathcal{X}$.

(3) *Preimage.* Let $\mathcal{W}$ be any set of $\mathcal{Y}$. The *pre-image* of $\mathcal{W}$ under the map A is

$$A^{-1}\mathcal{W} := \{x \in \mathcal{X} : Ax \in \mathcal{W}\}.$$

Observe that A need not be invertible, so beware of the distinction between preimage and inverse.

(4) *A-invariant subspace.* A subspace $\mathcal{V}$ of $\mathcal{X}$ is *A-invariant* if $A\mathcal{V} \subseteq \mathcal{V}$.

(5) *Reachable subspace.* Given the pair of conformable matrices $A_{n\times n}$ and $B_{n\times k}$, the *reachable subspace* of (A, B), denoted by $\langle A| \operatorname{Im} B\rangle$, is defined as $\langle A| \operatorname{Im} B\rangle := \operatorname{Im} \Gamma$, where Γ is the reachability matrix $[B \, AB \, \ldots \, A^{n-1}B]$. This is an $n \times nk$ matrix. The reachable subspace is A-invariant.

(6) $\langle A| \operatorname{Im} E\rangle$. The minimal A-invariant subspace that contains the subspace $\operatorname{Im} E$ is denoted by $\langle A| \operatorname{Im} E\rangle$.

(7) *Reachability subspace.* Given a matrix pair (A, B), a subspace $\mathcal{R}$ is called a *reachability subspace* if there are matrices F and G such that

$$\mathcal{R} = \langle A + BF| \operatorname{Im} BG\rangle.$$

(8) *Unobservable subspace.* Given the matrix pair (C, A), the *unobservable subspace* is defined as ker Ω, where Ω is the observability matrix

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

The unobservable subspace is A-invariant.

(9) The map A is *injective* if $Ax_1 = Ax_2$ implies $x_1 = x_2$.

(10) If $A : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{V} \subseteq \mathcal{X}$, the *restriction* of A to $\mathcal{V}$ is denoted by $A|_{\mathcal{V}}$.

(11) Given the two finite-dimensional subspaces $\mathcal{V}$ and $\mathcal{W}$, the *vector sum* is defined as

$$\mathcal{V} + \mathcal{W} := \{v + w : v \in \mathcal{V}, \, w \in \mathcal{W}\}.$$

If $\mathcal{V}$ and $\mathcal{W}$ are linearly independent we write the sum as $\mathcal{V} \oplus \mathcal{W}$. Note that in these notes $\oplus$ does *not* denote orthogonal vector sum!

(12) *Hypersurface.* Suppose N is an open set in R^n. The set M is defined as

$$M = \{x \in N : \lambda_i(x) = 0, i = 1, \ldots, n - m\}$$

where λ_i are smooth functions.

If rank $\begin{bmatrix} \frac{\partial \lambda_1}{\partial x} \\ \vdots \\ \frac{\partial \lambda_{n-m}}{\partial x} \end{bmatrix} = n - m \; \forall x \in M$, then M is a (hyper)surface

of dimension m.

(13) *Lie derivative.* In local coordinates, Lie derivative is represented by

$$L_f \lambda := \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i} f_i,$$

where f is a vector and λ is a scalar function.

(14) *Lie bracket.* Lie bracket of the two vector fields f and g is defined according to the rule:

$$[f, g](\lambda) := L_f L_g \lambda - L_g L_f \lambda.$$

In local coordinates the expression of $[f, g]$ is given as

$$\frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g.$$