Formula Sheet for Optimal Control

Division of Optimization and Systems Theory
Royal Institute of Technology
10044 Stockholm, Sweden

2003

February 16, 2007

1 Dynamic Programming

1.1 Discrete Dynamic Programming

General multistage decision problem

$$\min \phi(x_N) + \Sigma_{k=0}^{N-1} f_0(k, x_k, u_k) \quad \text{subj. to} \quad \begin{cases} x_{k+1} = f(k, x_k, u_k) \\ x_0 \text{ given} \\ u_k \in U(k, x_k) \end{cases} \tag{1}$$

Introduce the optimal *cost-to-go* function

$$J^*(n, x) = \min \phi(x_N) + \Sigma_{k=n}^{N-1} f_0(k, x_k, u_k) \text{ subj. to } \begin{cases} x_{k+1} = f(k, x_k, u_k) \\ x_n = x \\ u_k \in U(k, x_k) \end{cases}$$

for $n = 0, \ldots, N-1$ and $J^*(N, x) = \phi(x)$. In particular, the optimal solution of (1) is $J^*(0, x_0)$.

Theorem 1. *Consider the backwards dynamic programming recursion*

$$\begin{aligned} J(N, x) &= \phi(x), \\ J(n, x) &= \min_{u \in U(n,x)} \{f_0(n, x, u) + J(n+1, f(n, x, u))\}, \quad n = N-1, N-2, \ldots, 0 \end{aligned}$$

Then

(a) $J^*(n, x) = J(n, x)$ *for all* $n = 0, \ldots, N$, $x \in X_n$.

(b) *The optimal feedback control in each stage is obtained as*

$$u_n^* = \mu(n, x) = \text{argmin}_{u \in U(n,x)} \{f_0(n, x, u) + J(n, f(n+1, x, u))\}.$$

1.2 Continuous Time Dynamic Programming

consider the optimal control problem

$$\min \phi(x(t_f)) + \int_{t_i}^{t_f} f_0(t, x(t), u(t))dt \quad \text{subj. to} \quad \begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \\ x(t_i) = x_i, \;\; u(t) \in U \end{cases} \tag{2}$$

where t_i and t_f are fixed initial and terminal times and x_i is a fixed initial point. The end point $x(t_f)$ is free and can take any value in $\mathbf{R}^n$. The control is a piecewise continuous function, which satisfies the constraint $u(t) \in U$, for $t \in [t_i, t_f]$.

We define the *optimal cost-to-go function* as (this is also called the *value function*)

$$J^*(t_0, x_0) = \min_{u(\cdot)} J(t_0, x_0, u(\cdot))$$

where the minimization is performed with respect to all admissible controls. This means in particular that the optimization problem (2) can be written

$$J^*(t_i, x_i) = \min_{u(\cdot)} J(t_i, x_i, u(\cdot)).$$

Proposition 1. *The optimal-cost-to-go satisfies the Dynamic Programming Equation*

$$J^*(t_0, x_0) = \min_{u(\cdot)} \left\{ \int_{t_0}^{t} f_0(s, x(s), u(s))ds + J^*(t, x(t)) \right\}.$$

Theorem 2. *Suppose*

(*i*) $V : [t_i, t_f] \times \boldsymbol{R}^n \to \boldsymbol{R}$ *is* C^1 *(in both arguments) and solves HJBE*

$$\begin{aligned} -\frac{\partial V}{\partial t}(t, x) &= \min_{u \in U} \left\{ f_0(t, x, u) + \frac{\partial V}{\partial x}(t, x)^T f(t, x, u) \right\} \\ V(t_f, x) &= \phi(x) \end{aligned} \tag{3}$$

(*ii*) $\mu(t, x) = \arg \min_{u \in U} \left\{ f_0(t, x, u) + \frac{\partial V}{\partial x}(t, x)^T f(t, x, u) \right\}$ *is admissible.*

Then

(*a*) $V(t, x) = J^*(t, x)$ *for all* $(t, x) \in [t_i, t_f] \times \boldsymbol{R}^n$.

(*b*) $\mu(t, x)$ *is the optimal feedback control law, i.e.* $u^*(t) = \mu(t, x(t))$.

For a given optimal control problem on the form (2) we take the following steps[1]

[1]The same optimization as in step 2 is a part of the conditions in PMP.

1. Define the Hamiltonian

$$H(t,x,u,\lambda) = f_0(t,x,u) + \lambda^T f(t,x,u).$$

   Here $\lambda \in \mathbf{R}^n$ is a parameter vector.

2. Optimize pointwise over u to obtain

$$\widetilde{\mu}(t,x,\lambda) = \arg\min_{u\in U} H(t,x,u,\lambda) = \arg\min_{u\in U} \left\{ f_0(t,x,u) + \lambda^T f(t,x,u) \right\}.$$

3. Solve the partial differential equation

$$-\frac{\partial V}{\partial t}(t,x) = H\left(t, x, \widetilde{\mu}(t,x,\lambda), \frac{\partial V}{\partial x}(t,x)\right)$$

   subject to the initial condition $V(t_f,x) = \phi(x)$.

Then $\mu(t,x) = \widetilde{\mu}(t,x,\frac{\partial V}{\partial x}(t,x))$ is the optimal feedback control law, i.e. $u^*(t) = \mu(t,x(t))$.

2 PMP

Autonomous Systems

We consider the optimization problem

$$\min \phi(x(t_f)) + \int_0^{t_f} f_0(x(t),u(t))dt \quad \text{subj. to} \quad \begin{cases} \dot{x}(t) = f(x(t),u(t)) \\ x(0) \in S_i,\ x(t_f) \in S_f \\ u(t) \in U,\ t_f \geq 0 \end{cases} \tag{4}$$

where $S_f = \{x : G(x) = 0\}$ and

$$G(x) = \begin{bmatrix} g_1(x) \\ \vdots \\ g_p(x) \end{bmatrix}.$$

We have the following optimality conditions for problem (4): (Note: We IGNORE the pathological case and USE $\lambda_0 = 1$.)
PMP: Autonomuous Systems: Define the Hamiltonian

$$H(x,u,\lambda) = f_0(x,u) + \lambda^T f(x,u)$$

Assume that $(x^*(t), u^*(t), t_f^*)$ is an optimal solution to (4). Then there exists an adjoint function $\lambda(\cdot)$ that satisfies the following conditions

(i) $\dot{\lambda}(t) = -H_x(x^*(t), u^*(t), \lambda(t))$

(ii) $H(x^*(t), u^*(t), \lambda(t)) = \min_{v \in U} H(x^*(t), v, \lambda(t)) = 0$ for all $t \in [0, t_f^*]$ $(*)$

(iii) $\lambda(0) \perp S_i$

(iv) $\lambda(t_f^*) - \nabla\phi(x^*(t_f^*)) \perp S_f$

Remark 1. Condition (iv) is equivalent to

$$(\lambda(t_f^*) - \lambda_0 \nabla\phi(x^*(t_f^*)))^T v = 0 \quad \text{for all } v \text{ s.t.} \quad \begin{bmatrix} \frac{\partial g_1(x^*(t_f^*))}{\partial x_1} & \cdots & \frac{\partial g_1(x^*(t_f^*))}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_p(x^*(t_f^*))}{\partial x_1} & \cdots & \frac{\partial g_p(x^*(t_f^*))}{\partial x_n} \end{bmatrix} v = 0$$

which also can be written $\lambda(t_f^*) - \lambda_0 \nabla\phi(x^*(t_f^*)) \perp S_f$. Another equivalent formulation of this transversality condition is

$$\begin{aligned} \lambda(t_f^*) &= \lambda_0 \nabla\phi(x^*(t_f^*)) + G_x(x(t_f^*))^T \nu \\ &= \lambda_0 \nabla\phi(x^*(t_f^*)) + \sum_{k=1}^{p} \nu_k \nabla g_k(x^*(t_f^*)) \end{aligned}$$

for some vector $\nu \in \mathbf{R}^p$.

Special Case 1: It is reasonable to assume that the terminal cost and the terminal manifold involve two disjoint set of states. For example, $\phi(x) = \phi(x_{p+1}, \ldots, x_n)$ and $g_k(x) = g_x(x_1, \ldots, x_p)$, $k = 1, \ldots, p$. Then the transversality condition reduces to

$$\begin{bmatrix} \lambda_{p+1}(t_f^*) \\ \vdots \\ \lambda_n(t_f^*) \end{bmatrix} = \begin{bmatrix} \frac{\partial\phi(x(t_f^*))}{\partial x_{p+1}} \\ \vdots \\ \frac{\partial\phi(x(t_f^*))}{\partial x_n} \end{bmatrix} \tag{5}$$

and the remaining variables $(\lambda_1(t_f^*), \ldots, \lambda_p(t_f^*))$ remain undetermined.
Special Case 2: If $S_i = \{x_i\}$ (a given point) then there is no constraint on $\lambda(0)$.
Special Case 3: If $S_f = \mathbf{R}^n$ then $\lambda(t_f) = \nabla\phi(x^*(t_f^*))$.
Special Case 4: If $S_f = \mathbf{R}^n$ and $\phi = 0$ then $\lambda(t_f^*) = 0$.
Special Case 5: If $S_f = \{x_f\}$ (a given point) and $\phi = 0$ then there is no constraint on $\lambda(t_f)$.
Special Case 6: If the final time is fixed then $(*)$ is replaced by $H(x^*(t), u^*(t), \lambda(t)) = \min\limits_{v \in U} H(x^*(t), v, \lambda(t)) = \mathit{const}$ for all $t \in [0, t_f]$.

Nonautonomous systems

We consider the optimization problem

$$\min \phi(t_f, x(t_f)) + \int_{t_i}^{t_f} f_0(t, x(t), u(t))dt \quad \text{subj. to} \quad \begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \\ x(t_i) = x_i, \ x(t_f) \in S_f(t_f) \\ u(t) \in U, \ t_f \geq t_i \end{cases} \tag{6}$$

where the terminal manifold may depend on time:

$$S_f(t) = \{x \in \mathbf{R}^n : G(t, x) = 0\} \quad \text{where} \quad G(t, x) = \begin{bmatrix} g_1(t, x) \\ \vdots \\ g_p(t, x) \end{bmatrix}$$

and as usual we assume that the functional matrix

$$\begin{bmatrix} \frac{\partial g_1(x)}{\partial x_1} & \cdots & \frac{\partial g_1(x)}{\partial x_n} & \frac{\partial g_1(x)}{\partial t} \\ \vdots & & \vdots & \\ \frac{\partial g_p(x)}{\partial x_1} & \cdots & \frac{\partial g_p(x)}{\partial x_n} & \frac{\partial g_p(x)}{\partial t} \end{bmatrix}$$

has full rank.

We get the following optimality conditions for problem (6):(Note: We IGNORE the pathological case and USE $\lambda_0 = 1$.)

PMP: Nonautonomuous Systems: Define the Hamiltonian function

$$H(t, x, u, \lambda) = f_0(t, x, u) + \lambda^T f(t, x, u)$$

Assume that $(x^*(t), u^*(t), t_f^*)$ is an optimal solution to (6). Then there exists an adjoint function $\lambda(\cdot)$ that satisfies the following conditions

(*i*) $\dot{\lambda}(t) = -H_x(t, x^*(t), u^*(t), \lambda(t))$

(*ii*) $H^*(t) = \min_{v \in U} H(t, x^*(t), v, \lambda(t))$ satisfies

$$\begin{aligned} H^*(t) &= H^*(t_f^*) - \int_t^{t_f^*} \frac{\partial H}{\partial s}(s, x^*(s), u^*(s), \lambda(s))ds, \quad t \in [t_i, t_f^*] \\ H^*(t_f^*) &= -\sum_{k=1}^p \nu_k \frac{\partial g_k}{\partial t}(t_f^*, x^*(t_f^*)) - \frac{\partial \phi}{\partial t}(t_f^*, x^*(t_f^*)) \end{aligned} \tag{7}$$

(*iii*) $(\lambda(t_f^*) - \phi_x(t_f^*, x^*(t_f^*)) \perp S_f(t_f^*)$, which means that there must exist a vector $\nu = \begin{bmatrix} \nu_1 & \ldots & \nu_p \end{bmatrix}^T$ such that

$$\lambda(t_f^*) = \sum_{k=1}^p \nu_k \frac{\partial g_k}{\partial x}(t_f^*, x^*(t_f^*)) + \frac{\partial \phi}{\partial x}(t_f^*, x^*(t_f^*))$$

Special Case: If the terminal time is fixed then we can remove the time dependence of ϕ and S_f, i.e., the g_k are now only functions of the state. Conditions (ii) and (iii) are then replaced by

(ii) $H^*(t) = \min_{v \in U} H(t, x^*(t), v, \lambda(t))$ satisfies

$$H^*(t) = H^*(t_f) - \int_t^{t_f} \frac{\partial H}{\partial t}(s, x^*(s), u^*(s), \lambda(s) ds, \quad t \in [t_i, t_f]$$

(ii) $\lambda(t_f) - \nabla\phi(x^*(t_f)) \perp S_f$ or equivalently

$$\lambda(t_f) = \sum_{k=1}^{p} \nu_k \frac{\partial g_k}{\partial x}(x^*(t_f)) + \frac{\partial \phi}{\partial x}(x^*(t_f))$$

for some suitable vector $\nu = \begin{bmatrix} \nu_1 & \dots & \nu_p \end{bmatrix}^T$.

3 How to USE PMP

A professional way to address optimal control problems is to start investigating the vector field and the cost function to determine if

- it is possible to conclude that there must exist an optimal solution,
- the optimal solution is unique (generally hard).

The next step (in our case it would be the first) is to use PMP. We take the following steps (we consider problem (6) and assume $\lambda_0 = 1$)

1. Define the Hamiltonian: $H(t, x, u, \lambda) = f_0(t, x, u) + \lambda^T f(t, x, u)$
2. Perform pointwise minimization: $\widetilde{\mu}(t, x, \lambda) = \text{argmin}_{u \in U} H(t, x, u, \lambda)$, which means that a candidate optimal control is $u^*(t) = \mu(t, x(t), \lambda(t))$.
3. Solve the Two Point Boundary Value Problem (TPBVP)

   $$\begin{aligned} \dot{\lambda}(t) &= -H_x(t, x(t), \widetilde{\mu}(t, x(t), \lambda(t)), \lambda(t)), \quad \lambda(t_f) - \frac{\partial \phi}{\partial x}(t_f, x(t_f)) \perp S_f(t_f) \\ \dot{x}(t) &= H_\lambda(t, x(t), \widetilde{\mu}(t, x(t), \lambda(t)), \quad x(t_i) = x_i, \quad x(t_f) \in S_f(t_f) \end{aligned}$$

   One of the difficulties when solving a TPBVP is to find appropriate boundary conditions for x and λ. In order to obtain conditions that help us find candidates for the optimal transition time we also use (7) or (*). Sometimes we can determine the unknown parameters by plugging a parameterized control into the cost function and then optimize with respect to the parameters. This is a finite dimensional optimization problem.
4. Compare the candidate solutions obtained using PMP.

4 Infinite Time Horizon Optimal Control

Consider the optimal control problem

$$\min \int_0^\infty f_0(x,u)dt \quad \text{subject to} \quad \begin{cases} \dot{x} = f(x,u) \\ x(0) = x_0, \ u(t) \in U(x) \end{cases} \tag{8}$$

We assume without loss of generality that we want to control the system to an equilibrium point at $(x,u) = (0,0)$. This means that we assume $f(0,0) = 0$. In order to obtain a finite cost we further need to assume $f_0(0,0) = 0$.

Definition 1. A function $V : \mathbf{R}^n \to \mathbf{R}$ is called *positive semi-definite* if $V(0) = 0$ and $V(x) \geq 0$ for all $x \in \mathbf{R}^n$. If it satisfies the stronger condition $V(x) > 0$ for all $x \neq 0$ then it is called *positive definite.* It is called radially unbounded if $V(x) \to \infty$ when $\|x\| \to \infty$.

Example 1. A quadratic form $V(x) = x^T P x$, where $P = P^T$, is positive definite (semi-definite) if $P > 0$ ($P \geq 0$), i.e., if all eigenvalues of P are positive (non-negative). It is radially unbounded if $P > 0$.

Assumption 1. We assume that f_0 is positive semi-definite and positive definite in u, i.e., $f_0(x,u) \geq 0$, $\forall (x,u) \in \mathbf{R}^{n \times m}$ and $f_0(x,u) > 0$ when $u \neq 0$.

Assumption 2. We will assume that the artificial output $h(x) = f_0(x,0)$ of the system $\dot{x} = f(x,0)$ is observable in the sense that $h(x(t)) = 0$ for all $t \geq 0$ implies that $x(t) = 0$ for all $t \geq 0$.

Let us now define the optimal cost-to-go function (value function) corresponding to (8)

$$J^*(x_0) = \min_{u(\cdot)} \int_0^\infty f_0(x,u)dt.$$

The value function is independent of time since the dynamics and cost function of (8) both are independent of time.

Theorem 3. *Suppose Assumption 1 and Assumption 2 hold and*

(i) $V \in C^1$ *is positive definite, radially unbounded, and satisfies the (infinite horizon) HJBE*

$$\min_{u \in U} \left\{ f_0(x,u) + \frac{\partial V}{\partial x}(x)^T f(x,u) \right\} = 0 \tag{9}$$

(ii) $\mu(x) = \operatorname{argmin}_{u \in U} \left\{ f_0(x,u) + \frac{\partial V}{\partial x}(x)^T f(x,u) \right\}$.

Then

(*a*) $V(x) = J^*(x)$

(*b*) $u = \mu(x)$ *is an optimal globally asymptotically stabilizing feedback control.*

We next consider the special case of linear quadratic optimal control

Theorem 4. *Consider*

$$J^*(x_0) = \min \int_0^\infty [x^T Q x + u^T R u] dt$$

$$\text{subject to} \quad \begin{cases} \dot{x} = Ax + Bu \\ x(0) = x_0 \end{cases}$$

where $Q = C^T C$ and $R > 0$. We assume that (C, A) is observable and that (A, B) is controllable. Then

(*a*) $J^*(x_0) = x_0^T P x_0$, *where P is symmetric and positive definite ($P = P^T > 0$) is the unique positive definite solution to the Algebraic Riccati Equation (ARE)*

$$A^T P + PA + Q = PBR^{-1}B^T P. \tag{10}$$

(*b*) $\mu(x) = -R^{-1}B^T P x$ *is the optimal, stabilizing, feedback control.*

Remark 2. Conclusion (*b*) in particular means that the closed loop system matrix $A - BR^{-1}B^T P$ has all eigenvalues in the open left half plane.

The linear quadratic regulator in Theorem 4 satisfies certain robustness properties. The following inequality derived from the ARE is of key importance

Proposition 2. *Let $L = R^{-1}B^T P$, where P is a solution to the ARE in (10). Then the transfer function*

$$G(s) = L(sI - A)^{-1}B$$

satisfies the inequality

$$(I + G(j\omega))^* R (I + G(j\omega)) \geq R \tag{11}$$

5 Second order variations

Consider

$$\min \phi(x(t_f)) + \int_0^{t_f} f_0(t, x(t), u(t)) dt \quad \text{subj to} \quad \begin{cases} \dot{x} = f(t, x(t), u(t)) \\ x(0) = x_0, \end{cases} \tag{12}$$

where ϕ, f_0, and f are twice continuously differentiable with respect to x and u.

Proposition 3. *Suppose* $(x^*(\cdot), u^*(\cdot))$*, and* $\lambda(\cdot)$ *are such that*

(i) $\dot{x}^*(t) = f(t, x^*(t), u(t)),\ x^*(0) = x_0,$

(iia) $\dot{\lambda}(t) = -H_x(t, x^*(t), u^*(t), \lambda(t)),\ \lambda(t_f) = \phi_x(x^*(t_f))$

(iib) $H_u(t, x^*(t), u^*(t), \lambda(t)) = 0$

(iiia) $\phi_{xx}(x^*(t_f)) \geq 0$

(iiib) $H^*_{uu}(t) > 0$ *and* $\begin{bmatrix} H^*_{xx}(t) & H^*_{xu}(t) \\ H^*_{ux}(t) & H^*_{uu}(t) \end{bmatrix} \geq 0$*, where* $H^*_{uu}(t) = H_{uu}(t, x^*(t), u^*(t), \lambda(t))$ *and similarly for* H^*_{xx}, H^*_{xu} *and* H^*_{ux}.

Then $(x^*(\cdot), u^*(\cdot))$ *is a local minimum of (12).*