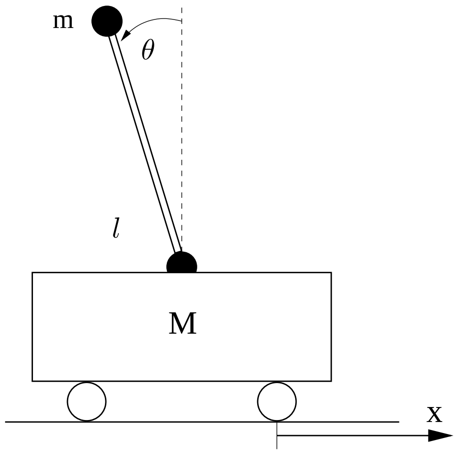
KTH Matematik

Homework 2: 5B1873: Optimal Control Spring 2007
Grading: You may use $\min(1, \frac{\text{your credit}}{30})$ extra points on the exam.
The homework is due on February 19, 2007 at 17.00

1. Consider the cart and pendulum system in Figure 1. Derive a dynamical equation for the system using Euler Lagranges equation (see the example in Chapter 5).

.... (10p)



Figur 1: An inverted pendulum mounted on a cart

2. The dynamics of mobile robots can under some simplifying assumptions be linearized using feedback. The resulting model becomes

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{1}$$

where $y = (y_1, y_2)$ denotes the position of the robot and $u = (u_1, u_2)$ is the control signal. We assume that the robot is initially at rest at the origin, i.e. $y(0) = \dot{y}(0) = (0, 0)$.

We want to design the control signal such that the robot tracks the circular motion

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

on the time interval $[0, 8\pi]$. One way to do this is to solve the following optimization problem

$$\min \int_0^{8\pi} [(y_1(t) - \cos(t))^2 + (y_2(t) - \sin(t))^2 + r(u_1(t)^2 + u_2(t)^2)]dt$$

subject to the linear dynamics

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2}$$

Use Matlab to solve the problem. Choose r such that $|u_k(t)| \leq 5$ for $k = 1, 2$ and such that good tracking is obtained. Plot your solutions $y_1(t), y_2(t)$ and $u(t)$. Also plot $(y_1(t), y_2(t))$, i.e. y_2 as a function of y_1. Include the Matlab code in the solution you hand in. Note that you cannot obtain perfect tracking.

.... (10p)

3. In this example we consider a control problem for the rockett car in Figure 2. The car starts from rest at $z = -1$ and should be controlled such that after five seconds the car is close to be at rest at the origin (or at rest at the origin). If we define the state to be $x_1 = z$ and $x_2 = \dot{z}$ then the problem is to find a control $u : [0, 5] \to \mathbf{R}$ such that the solution to

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0 := \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

satisfies $x(5) = 0$ or at least $x(5) \approx 0$. Four control engineers have suggested the controllers $C1 - C4$ below.

$(C1)$ $u(t) = -5x_1(t) + D$

$(C2)$ $u(t) = -\frac{1}{1+5-t+(5-t)^3/3+(5-t)^4/12} \begin{bmatrix} 5 - t + \frac{(5-t)^2}{2} & 1 + (5-t)^2 + \frac{(5-t)^3}{3} \end{bmatrix} x(t) + D$

$(C3)$ $u(t) = -\begin{bmatrix} \frac{6}{(5-t)^2} & \frac{4}{5-t} \end{bmatrix} x(t) + D$

$(C4)$ $u(t) = \frac{12}{5^3}(\frac{5}{2} - t) + D.$

where D is a disturbance. You also know that some but not all of these engineers used optimal control to derive their controller. They have informed that each of $C1 - C4$ corresponds to either of $O1 - O3$. It is possible that two of the controllers corresponds to the same optimization problem.

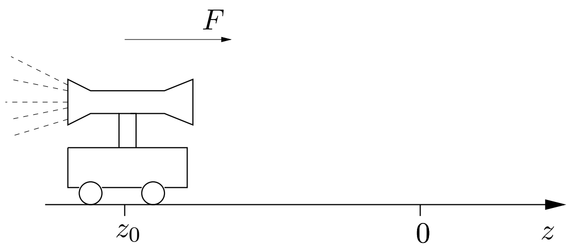
$(O1)$ $\min \int_0^5 u(t)^2 dt$ s.t. $\dot{x} = Ax + Bu,\ x(0) = x_0,\ x(5) = 0$

$(O2)$ $\min \|x(5)\|^2 + \int_0^5 u(t)^2 dt$ s.t. $\dot{x} = Ax + Bu,\ x(0) = x_0$

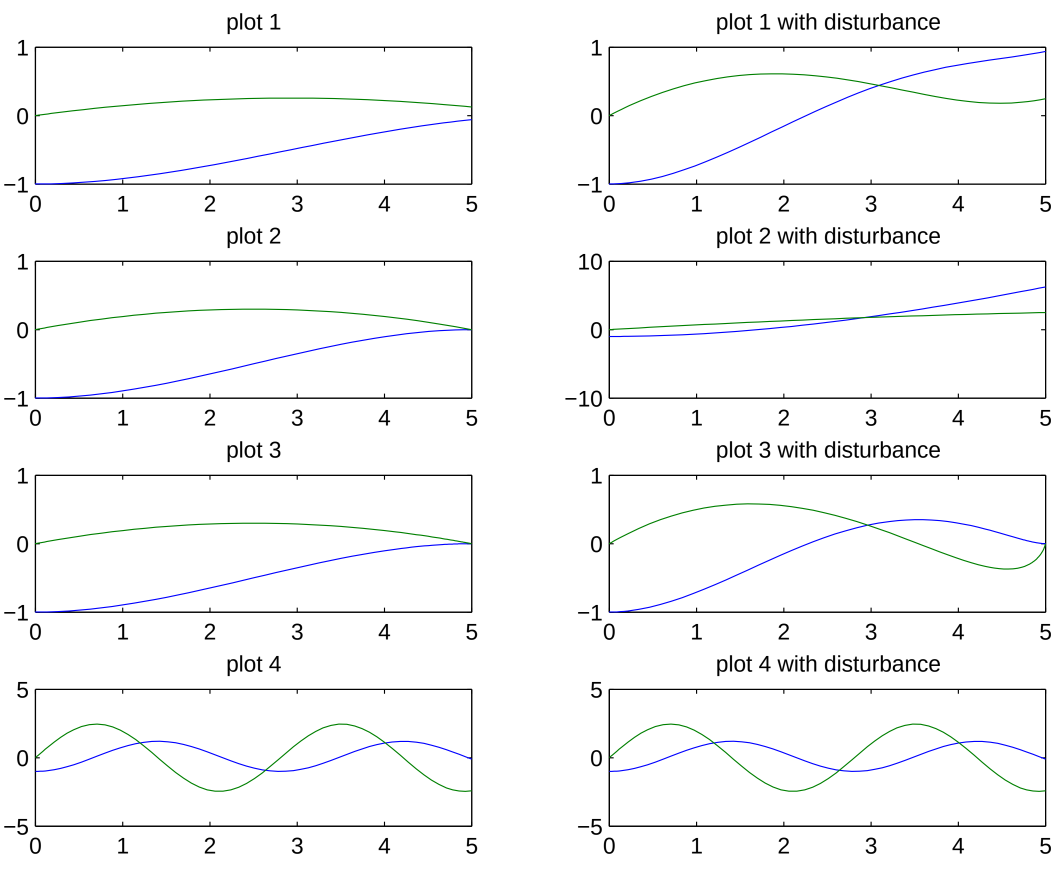
$(O3)$ None of these two optimal control problems.

The get a better understanding of the four controllers a fifth engineer performed the simulations in Figure 3. The left hand side plots corresponds to the case when $D = 0$ and the right hand side plots corresponds to $D = 0.5$. Unfortunately, the engineer did not tell you which plot corresponds to what controller.

Your problem is to match the controllers $C1 - C4$ with the plots 1-4 and for each controller determine which of the optimal control problems $O1 - O3$ was used for the design. You must motivate your answer. .... (10p)



Figur 2: Control problem: Move the rocket car to rest at $z = 0$.



Figur 3: The closed loop response for four different controllers. The left hand side figure shows the two states for the case with no disturbance. The right hand side figure shows the response in the case with disturbance.

4. Let $u^* : [t_0, t_f^*] \to U$ generate a solution $x^* : [t_0, t_f^*] \to \mathbf{R}^n$ to the boundary value problem

$$\dot{x} = f(x,u), \quad x(0) = x_0, \quad x(t_f) \in S_f \tag{3}$$

where $S_f = \{x \in \mathbf{R}^n : G(x) = 0\}$ is a smooth manifold. Suppose there exists $V : \mathbf{R}^n \setminus S_f \to \mathbf{R}$ of class[1] C^1 such that

(i) $f_0(x^*(t), u^*(t)) + V_x(x^*(t))^T f(x^*(t), u^*(t)) = 0$, $t \in [t_0, t_f^*)$

(ii) $f_0(x,u) + V_x(x)^T f(x,u) \geq 0$, $\forall x \in \mathbf{R}^n \setminus S_f$, $\forall u \in U$.

(iii) if $u : [t_0, t_f] \to U$ generates another solution $x(\cdot)$ to the boundary value problem (3) then

$$\lim_{t \to t_f} V(x(t)) \leq \lim_{t \to t_f^*} V(x^*(t)) = 0$$

From $(i) - (iii)$ and (3) we can derive the following relationship

$$\begin{aligned}
&\lim_{t \to t_f} \int_{t_0}^{t} f_0(x(t), u(t))dt - \lim_{t \to t_f^*} \int_{t_0}^{t} f_0(x^*(t), u^*(t))dt \\
&\geq V(x(t_0)) - \lim_{t \to t_f} V(x(t)) - (V(x^*(t_0)) - \lim_{t \to t_f^*} V(x^*(t))) = - \lim_{t \to t_f} V(x(t)) \geq 0
\end{aligned}$$

(a) Explain each step of the derivation above. ....(4p)

(b) What conclusion can you draw about the control function $u^*(\cdot)$?. ....(3p)

(c) Apply the above theory to the case when $u^*(t) = -\text{sign}(x_0)$, $f_0(x,u) = 1$, $f(x,u) = -x + u$, $t_f^* = V(x_0) = \ln(1 + \text{sign}(x_0)x_0)$. Interpret in terms of an optimal control problem. ....(3p)

[1] This means that V is continuously differentiable everywhere except at points in S_f.