KTH Matematik

Homework 3: 5B1873: Optimal Control Spring 2007

Grading: You may use $\min(1, \frac{\text{your credit}}{30})$ extra points on the exam.

The homework is due on March 1, 2006 at 17.00

- Note that there are more problems than you need to solve to get full score. You are allowed to work on all problems.
- Please answer the questionaire attached to the homework set. You may hand it in with your homework set or put it in the black mailbox on the first floor of the mathematics building.

1. Linear time-varying systems appears frequently in optimal control, for example when solving the adjoint equation in PMP. Let $\Phi(t,s)$ denote the transition matrix corresponding to $A(t)$.

Indicate whether the following statements are **true** or **false**

( ) $\Phi(t_2,t_1)\Phi(t_1,t_2) = I$

( ) $\Phi(t_4,t_3)\Phi(t_3,t_2)\Phi(t_2,t_1) = \Phi(t_1,t_4)$

( ) $\dot{x}(t) = -A(t)^T x(t)$, $x(t_f) = x_f$ has the solution $x(t) = \Phi(t_f,t)^T x_f$

( ) $\dot{x}(t) = \cos(t)x(t)$, $x(\pi) = x_0$ has the solution $x(t) = e^{\sin(t)}x_0$.

( ) $\dot{x}(t) = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} x(t)$ has the solution $x(t) = \begin{bmatrix} e^{at} & e^{at}t \\ 0 & e^{at} \end{bmatrix} x(t_0)$

All your answers should be shotly motivated.

You get one point credit for every correct answer and minus one point credit for wrong ansers. Your total score will always be nonnegative. ....(5)

2. Determine the boundary conditions on the adjoint vector, $\lambda(t_f)$, for the following optimal control problem

$$\max_{u(\cdot)} x_1(1) \quad \text{s.t.} \quad \begin{cases} \dot{x}_1(t) = c_1 x_2(t), & x_1(0) = x_{10} \\ \dot{x}_2(t) = \dfrac{x_3(t)^2}{x_1(t)} - \dfrac{1}{x_1(t)^2} + \dfrac{\sin(u(t))}{1/c_2 - c_3 t}, & x_2(0) = x_{20} \\ \dot{x}_3(t) = -\dfrac{c_4 x_2(t) x_3(t)}{x_1(t)} + \dfrac{c_5 \cos(u(t))}{1/c_1 - c_3 t}, & x_3(0) = x_{30} \\ x(1) \in S_f \end{cases}$$

where $S_f = \left\{ x \in \mathbf{R}^3 : x_2 = 0,\ x_3 - \frac{1}{\sqrt{x_1}} = 0 \right\}$.

.... (4p)

3. Write a short exposition evaluating the relative merits of dynamic programming and the Pontryagin minimum principle for solving optimal control problems in continuous time. List some advantages and disadvantages of the two methods.

.... (4p)

4. Solve the optimal control problem

$$\max \int_0^2 (1-u(t))x(t)dt \quad \text{subject to} \quad \begin{cases} \dot{x}(t) = u(t)x(t) - \frac{1}{3}x(t) \\ x(0) = a > 0 \\ 0 \le u(t) \le 1 \end{cases}$$

Hint: Note that $x(t)$ always will be positive.

.... (10p)

5. In this problem we will investigate a property of the value function for infinite-horizon linear quadratic optimal control problems of the general form

$$V(x_0) = \min_{u(\cdot)} \int_0^\infty (x^T Qx + u^T Ru)dt \quad \text{s.t.} \quad \dot{x} = Ax + Bu,\ x(0) = x_0$$

where $Q = C^T C$, (C, A) is observable, $R = R^T > 0$.

(*a*) If $c = 1$, $r = 1$, $A = 1$, $B = 1$, then $p = 1 + \sqrt{2}$ and the feedback law $u = -(1+\sqrt{2})x$. Then $V(x(t)) = px(t)^2$ is positive definite. Is the time derivative $\frac{1}{dt}V(x(t))$ postive or negative (semi) definite?

.... (3p)

(*b*) In the general case $V(x(t)) = x(t)^T Px(t)$ where P is the positive definite solution to the Riccati equation

$$A^T P + PA + Q - PBR^{-1}B^T P = 0$$

and the feedback control is $u = -R^{-1}B^T Px$. Make a conjecture about the definiteness of the time derivative of $V(x(t))$.

.... (2p)

(*c*) Prove your conjecture.

.... (5p)

6. We consider the optimization problem

$$\min \phi(t_f, x(t_f)) + \int_{t_i}^{t_f} f_0(t, x(t), u(t))dt \quad \text{subj. to} \quad \begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \\ x(t_i) = x_i,\ x(t_f) \in S_f(t_f) \\ u(t) \in U,\ t_f \ge t_i \end{cases} \tag{1}$$

where $\phi(t,x)$ is assumed to be continuously differentiable with respect to both arguments, $f_0(t,x,u)$ and $f(t,x,u)$ are continuously differentiable with respect to t and x, and the terminal manifold may depend on time:

$$S_f(t) = \{x \in \mathbf{R}^n : G(t,x) = 0\} \quad \text{where} \quad G(t,x) = \begin{bmatrix} g_1(t,x) \\ \vdots \\ g_p(t,x) \end{bmatrix}$$

It is assumed that the functional matrix

$$\begin{bmatrix} \frac{\partial g_1(x)}{\partial x_1} & \cdots & \frac{\partial g_1(x)}{\partial x_n} & \frac{\partial g_1(x)}{\partial t} \\ \vdots & & \vdots & \\ \frac{\partial g_p(x)}{\partial x_1} & \cdots & \frac{\partial g_p(x)}{\partial x_n} & \frac{\partial g_p(x)}{\partial t} \end{bmatrix}$$

has full rank.

Use Pontrygin's minimum principle to derive necessary conditions for optimality of this optimal control problem.

. . . . (10p)

Hint: In the lecture notes there are some hints on how to proceed. In particular, you should introduce the new state $x_{n+1}(t) = t$.

Questionaire

We would like to have your opinion on the different homework problems in the course. We appreciate if you take some time to fill out this questionaire.

1. Comprehension of definitions and procedures (HW3: Problem 1 and 2)

	Very easy	Easy	Reasonable	Quite hard	Too Hard
I would rate the questions as					
	Very clear	Clear	Reasonable	Quite hard	Too hard
I understand the questions					
	Very much	A lot	Yes	Not much	Nothing
I learnt from these problems					

Further comments on these problems: ....

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2. Application of procedures and algorithms (HW1: Problem 1, HW2: Problem 1, HW3: Problem 4)

	Very easy	Easy	Reasonable	Quite hard	Too Hard
I would rate the questions as					
	Very clear	Clear	Reasonable	Quite hard	Too hard
I understand the questions					
	Very much	A lot	Yes	Not much	Nothing
I learnt from these problems					

Further comments on these problems: ....

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3. Problems that involve modeling and application of algorithms in new situations. HW1 problem 2.

	Very easy	Easy	Reasonable	Quite hard	Too Hard
I would rate the question as					
	Very clear	Clear	Reasonable	Quite hard	Too hard
I understand the questions					
	Very much	A lot	Yes	Not much	Nothing
I learnt from this problem					

Further comments on these problems: ....

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4. Applications using Matlab. HW1: Problem 3, HW2: Problem 2.

	Very easy	Easy	Reasonable	Quite hard	Too Hard
I would rate the questions as					
	Very clear	Clear	Reasonable	Quite hard	Too hard
I understand the questions					
	Very much	A lot	Yes	Not much	Nothing
I learnt from these problems					

Further comments on these problems: ....

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5. Comparison of results. HW 2: Problem 3.

	Very easy	Easy	Reasonable	Quite hard	Too Hard
I would rate the question as					
	Very clear	Clear	Reasonable	Quite hard	Too hard
I understand the question					
	Very much	A lot	Yes	Not much	Nothing
I learnt from this problem					

Further comments on this problem: ....

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6. Theoretical problems

(a) Proof of new situation by extrapolation of known result. HW3: Problem 6.

(b) Conjecture and prove results: HW3: Problem 5.

(c) Explain proof and make conclusions and interpretation: HW2: Problem 4.

	Very easy	Easy	Reasonable	Quite hard	Too Hard
I would rate the questions as					
	Very clear	Clear	Reasonable	Quite hard	Too hard
I understand the questions					
	Very much	A lot	Yes	Not much	Nothing
I learnt from these problems					

Can you distinguish any difference between the three problems. Do you have any preferences?

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Further comments on these problems: ....

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