Optimal Control to a Manifold

$$J^*(x_0) = \min \int_0^{t_f} f_0(x,u)dt \quad \text{s.t.} \quad \begin{cases} \dot{x} = f(x,u), \quad x(0) = x_0 \\ x(t_f) \in S_f = \{x : G(x) = 0\} \\ u \in U, \ t_f > 0 \end{cases}$$

Note that the terminal time is a parameter to be optimized, which is the reason why J^* is a function of the initital state but not the initial time. Try to prove this!

The Hamilton-Jacobi-Bellman equation becomes

$$0 = \min \left\{ f(x,u) + V_x(x)^T f(x,u) \right\}$$
$$V(x) = 0, \quad \text{when} \quad G(x) = 0$$

Application to Time-Optimal Control

$$J^*(x_0) = \min t_f \quad \text{when} \quad \begin{cases} \dot{x} = -x + u, \quad x(0) = x_0 \\ x(t_f) = 0 \\ |u| \le 1, \quad t_f > 0 \end{cases}$$

It is easy to verify that

$$u^*(t) = -\text{sign}(x(t))$$
$$J^*(x) = \ln(1 + \text{sign}(x)x)$$

We note that $J^*(x)$ satisfies HJBE

$$0 = 1 - V'(x)x - |V(x)|, \text{ when } x \neq 0 \text{ and } V(0) = 0.$$

However, $J^* \notin C^1$ so our theory is not valid for this example. More on this in the next homework.

An LQ problem with no solution

$$\min \int_0^T (-x(t)^2 + u(t)^2)dt \quad \text{s.t.} \quad \dot{x} = u, \quad x(0) = x_0$$

The Riccati equation for this LQ problem is

$$\dot{p} = 1 + p^2, \quad p(0) = 0$$

The solution becomes

$$p(t) = \tan(t - T)$$

Note if $T > \pi/2$ then the solution $p(t)$ seases to exist at $t = T - \pi/2$ (finite escape time)

An Existence and Uniqueness Result

Theorem 1. *The Riccati equation*

$$\dot{P} + A^T P + PA + Q = PBR^{-1}B^T P, \quad P(T) = Q_0$$

where $Q = Q^T \ge 0$, $Q_0 = Q_0^T \ge 0$, $R = R^T > 0$ *has a unique solution on* $[0, T]$.