Feedback Versus Open Loop

January 31, 2007

1 Control over a finite time-horizon

Consider a particle moving along a 1-dimensional axis

$$x_{k+1} = x_k + u_k, \quad x_0 \text{ given.}$$

The state x denotes the position of the particle and the control u is the movement of the particle from one time instance to the next. We will compare open loop and feedback solutions for this problem with respect to their ability to reduce the effect of disturbances.

Let us assume that we want to bring the particle to the origin in two steps while using minimum control energy. This gives rise to the optimization problem

$$\begin{aligned} J^*(0, x_0) &= \min_u u_0^2 + u_1^2 \quad \text{s.t.} \quad \begin{cases} x_{k+1} = x_k + u_k \\ x_2 = 0 \end{cases} \\ &= \min u_0^2 + u_1^2 \quad \text{s.t.} \quad \mathrm{x}_0 + \mathrm{u}_0 + \mathrm{u}_1 = 0 \\ &= \min u_0^2 + (x_0 + u_0)^2 = \frac{1}{2} x_0^2 \end{aligned}$$

and the optimal control sequence becomes

$$\begin{aligned} u_0^* &= \mu(0, x_0) = -\frac{1}{2} x_0, \\ u_1^* &= \mu(1, x_0) = -\frac{1}{2} x_0. \end{aligned}$$

The notation $u_k^* = \mu(k, x_0)$ is used to clarify that the control depends on the time k and the initial state x_0. Such control laws are called *open loop control.*

We obtain an alternative solution by using dynamic programming. The DynP algorithm gives

$$J(2, x) = \begin{cases} 0, & x = 0 \\ \infty, & x \neq 0 \end{cases}$$

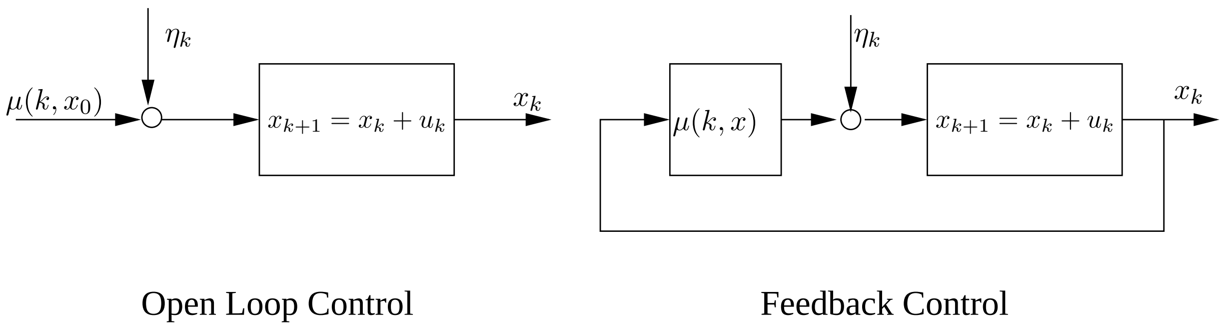

Figure 1: Open loop versus feedback control.

Note that the cost is infinite unless the constraint $x = 0$ is satisfied. The next step of the DynP algorithm gives

$$J(1, x) = \min_u \left\{ u^2 + J(1, x + u) \right\} = x^2$$

and the minimizing control is $u^* = \mu(1, x) = -x$. We used that $u = -x$ because otherwise $J(1, x + u) = \infty$. The final step of the DynP algorithm gives

$$J(0, x) = \min_u \left\{ u^2 + J(0, x + u) \right\} = \min_u \left\{ u^2 + (x + u)^2 \right\} = \frac{1}{2} x^2$$

and the minimizing control is $u^* = \mu(0, x) = -\frac{1}{2} x$. Hence the optimal control is

$$u_0^* = \mu(0, x) = -\frac{1}{2} x$$
$$u_1^* = \mu(1, x) = -x$$

The notation $u_k^* = \mu(k, x_k)$ is used to clarify that the control depends on both time and the *current* state. Such control laws are called *feedback control.*

1.1 Disturbance Sensitivity

Consider the situation in Fig. 1 where the left part illustrates the open loop situation and the right hand side illustrates the feedback control situation. The signal η_k denotes a disturbance and we first assume $\eta_0 \neq 0$ and $\eta_k = 0$, $k \geq 1$.

For the open loop situation we get the situation

$$x_1 = x_0 + \mu(0, x_0) + \eta_0 = \frac{1}{2} x_0 + \eta_0$$
$$x_2 = x_1 + \mu(1, x_0) = \eta_0$$

We do not reach the origin as desired!

In the closed loop situation we get

$$x_1 = x_0 + \mu(0, x_0) + \eta_0 = \frac{1}{2}x_0 + \eta_0$$
$$x_2 = x_1 + \mu(1, x_1) = 0$$

The feedback compensated for the disturbance. Note however, that if the disturbance is persistent, i.e. $\eta_k \neq 0$, $k \geq 1$, then the position of the particle can still be disturbed and thus deviate from the desired position at the origin. In order to avoid such a situation we could consider optimal control over an infinite time horizon.

1.2 Infinite Horizon Optimal Control

Consider

$$\min_u \sum_{k=1}^{\infty} x_k^2 + u_k^2 \quad \text{s.t.} \quad x_{k+1} = x_k + u_k$$

The cost function forces the state and the control to converge to zero. We obtain a solution by solving the Bellman equation

$$J(x) = \min_u \left\{x^2 + u^2 + J(x+u)\right\}$$

Let us try the form $J(x) = px^2$, where $p > 0$ in order for J to be positive definite. This gives

$$\begin{aligned} px^2 &= \min_u \left\{x^2 + u^2 + p(x+u)^2\right\} \\ &= \min_u (1+p)(u + \frac{p}{1+p}x)^2 + x^2(1 + p - \frac{p^2}{1+p}) \\ &= x^2(1 + p - \frac{p^2}{1+p}) \end{aligned}$$

Hence, the optimal feedback control is

$$u^* = \mu(x) = -\frac{p}{1+p}x$$

where p is the positive solution to the Riccati equation

$$p^2 = 1 + p \tag{1}$$

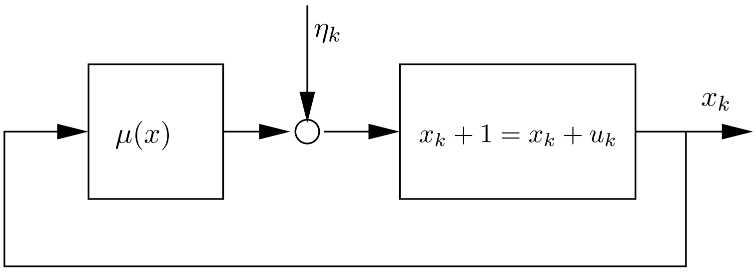

Figure 2: Infinite horizon control.

Let us consider the situation in Fig 2. We obtain the solution

$$
\begin{aligned}
x_{k+1} &= x_k + \mu(x_k) + \eta_k = \frac{1}{1+p} x_k + \eta_k \\
&= \frac{1}{1+p}(x_{k-1} + \mu(x_{k-1}) + \eta_{k-1}) + \eta_k \\
&= \frac{1}{(1+p)^2} x_{k-1} + \frac{1}{1+p}\eta_{k-1}) + \eta_k \\
&= \ldots = \frac{1}{(1+p)^{k+1}} x_k + \sum_{l=0}^{k} \frac{1}{(1+p)^{k-l}} \eta_l
\end{aligned}
$$

We see that the influence of the initial condition decays to zero since $1/(1+p) < 1$. We also see that the contribution from old disturbances is reduced as time evolves. Hence, the infinite time horizon optimal control problem results in a stabilizing (convergence) feedback controller that also gives robustness to the disturbance.

In summary:

- Feedback solutions have the advantage that the effect of disturbances can be compensated for.
- Infinite time horizon optimal control gives both convergence and disturbance compensation. Note that the convergence to the desired value (zero in our example) in general is slower than if a finite time horizon optimal criterion is used for the control design.
- It is usually easier to derive an open loop controller than a feedback controller.