Infinite Time Horizon Linear Quadratic Control

1. Stability definitions
2. Positive definiteness
3. The infinite time horizon LQ control theorem

Stability definitions

Consider

$$\dot{x} = f(x), \qquad x(0) = x_0, \tag{1}$$

Definition 1. *$x = 0$ is an equilibrium point for (1) if $f(0) = 0$,*

Definition 2. *$x = 0$ is a stable equilibrium of (1) if $\forall \epsilon > 0$, $\exists \delta(\epsilon) > 0$ such that*

$$\|x_0\| \leq \delta(\epsilon) \quad \Rightarrow \quad \|x(t)\| \leq \epsilon, \qquad t \geq 0$$

Definition 3. *$x = 0$ is an asymptotically stable equilibrium of (1) if*

(i) *it is a stable equilibrium*

(ii) *$x(t) \to 0$ as $t \to \infty$*

Positive Definiteness

Definition 4. *A function $V : \mathbf{R}^n \to \mathbf{R}$ is positive semi-definite if*

$$V(x) \geq 0 \quad \text{and} \quad V(0) = 0$$

and it is a positive definite function if

$$V(x) > 0,\ x \neq 0 \quad \text{and} \quad V(0) = 0$$

It is radially unbounded if $V(x) \to \infty$ if $\|x\| \to \infty$

Example 1. *$V(x) = x^T Q x$ is*

- *positive semi-definite if Q is positive semi-definite (denoted $Q \geq 0$), i.e. $\lambda(Q) \geq 0$ for all eigenvalues of Q.*
- *positive definite if Q is positive definite (denoted $Q > 0$), i.e. $\lambda(Q) > 0$ for all eigenvalues of Q.*

The Infinite Time Horizon LQ control theorem

$$\min_{u \in \mathcal{U}} \underbrace{\int_0^\infty (x(t)^T Q x(t) + u(t)^T R u(t))dt}_{J(u)} \tag{2}$$

$$\text{subj. to} \quad \dot{x}(t) = Ax(t) + Bu(t),\ x(t_0) = x_0$$

$$\mathcal{U} = \left\{ u : \int_0^\infty (\|x\|^2 + \|u\|^2)dt < \infty \right\}$$

- Stability guarantee due to the constraint $u \in \mathcal{U}$
- Performance guarantee due to the cost integral

Theorem 1. *Suppose*

(i) $R > 0$ *and* $Q = C^T C$

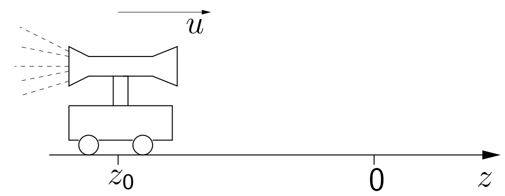
(ii) (A, B, C) *is a minimal realisation.*

The solution to (2) is given by the state feedback $u(t) = -R^{-1}B^T P x(t)$, *where* $P = P^T$ *is the unique positive definite solution to the ARE*

$$A^T P + PA + Q - PBR^{-1}B^T P = 0 \tag{3}$$

The optimal cost is $\min_{u \in \mathcal{U}} J(u) = x_0^T P x_0$.

The Rocket Car



Drive the rocket car from rest at position z_0 to approximately rest at position 0. LQ optimal control provides three alternatives

- Fixed end point control with minimum energy
- Finite time horizon LQ optimal control with terminal cost
- Infinite time horizon LQ optimal control

Infinite Time Horizon LQ

$$\min_u \int_0^\infty (\|x\|^2 + \|u\|^2)dt \quad \text{subj. to} \quad \dot{x} = Ax + Bu, \; x(0) = x_0$$

The ARE $A^T P + PA + I - PBB^T P = 0$ becomes

$$\begin{bmatrix} -p_{12}^2 + 1 & -p_{12}p_{22} + p_{11} \\ -p_{12}p_{22} + p_{11} & 2p_{12} - p_{22}^2 + 1 \end{bmatrix} = 0$$

which implies

$$P_{12} = \pm 1$$
$$P_{22} = P_{11} = \pm\sqrt{1 \pm 2}$$

The positive definite solution to the ARE is

$$P = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix}$$

The optimal state feedback is $u = -R^{-1}B^T P x = -\begin{bmatrix} 1 & \sqrt{3} \end{bmatrix} x$ and the closed loop system matrix becomes

$$A - BB^T P = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{3} \end{bmatrix}$$

which is a stable matrix.

```
>> H=[0 1 0 0;0 0 0 -1;-1 0 0 0;0 -1 -1 0];
>> [V D]=eig(H)
V =
   0.5000              0.5000              0.4330 - 0.2500i   0.4330 + 0.2500i
   0.4330 + 0.2500i   0.4330 - 0.2500i  -0.2500 + 0.4330i  -0.2500 - 0.4330i
  -0.4330 + 0.2500i  -0.4330 - 0.2500i   0.5000             0.5000
  -0.2500 - 0.4330i  -0.2500 + 0.4330i  -0.0000 + 0.5000i  -0.0000 - 0.5000i

D =
   0.8660 + 0.5000i        0                  0                  0
        0             0.8660 - 0.5000i        0                  0
        0                  0             -0.8660 + 0.5000i        0
        0                  0                  0             -0.8660 - 0.5000i

>> X=V(1:2,3:4);
>> Y=V(3:4,3:4);
>> P=Y/X
P =
  1.7321 - 0.0000i   1.0000 - 0.0000i
   1.0000 - 0.0000i   1.7321
```