MPC Problems

January 31, 2007

Example 1. Consider a model predictive control problem where at each time instance the following optimization problem is solved: $\min_{U_t} J(x_{t|t}, U_t)$ where

$$J(x_{t|t}, U_t) = \sum_{k=0}^{1} x_{t+k|t}^T Q x_{t+k|t} + u_{t+k|t}^2$$

$$\text{subject to } \begin{cases} x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t} \\ -1 \leq Cx_{t+k|t} \leq 1,\ k = 0, 1 \\ x_{t+2|t} = 0, \quad -1 \leq u_{t+k|t} \leq 1,\ k = 0, 1 \end{cases}$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Determine the set of initial conditions for which the closed loop system is stable.
Solution: We use the stability result on MPC in the handout from ETH (Löfberg et. al.). We notice that

1. $f_0(x, u) = x^T Qx + u^2$ is a strictly positive definite function since $f_0(0, 0) = 0$ and

   $$f_0(x, u) \geq (\|x\|^2 + u^2).$$

   since $Q = I$.

2. $f(x, u) = Ax + Bu$ satisfies $f(0, 0) = 0$.

3. The state constraint set $\mathbf{X} = \{x : -1 \leq Cx \leq 1\}$ contains the origin.

4. The control constraint set $\mathbf{U} = \{u : -1 \leq u \leq 1\}$ contains the origin.

5. The terminal control constraint is $x_{t+2|t} = 0$.

Under these conditions it follows that the region of stability is $\mathbf{X}_{0,feas} = \{x : \min_{U_t} J(x, U_t) < \infty$. If the initial condition belongs to $\mathbf{X}_{0,feas}$ then the MPC algorithm converges to zero. $\mathbf{X}_{0,feas}$ is the set of states for which the constraints of the optimization problem is satisfied, i.e.

$$\begin{aligned} & -1 \le Cx \le 1 \\ & -1 \le CAx + CBu_0 \le 1 \\ & A^2x + ABu_0 + Bu_1 = 0 \\ & -1 \le u_k \le 1, \quad k = 0, 1 \end{aligned}$$

which is equivalent to

$$\begin{aligned} & -1 \le x_1 + x_2 \le 1 \\ & -1 \le x_1 + 2x_2 + u_0 \le 1 \\ & x_1 + 2x_2 + u_0 = 0 \\ & x_2 + u_0 + u_1 = 0 \\ & -1 \le u_k \le 1, \quad k = 0, 1 \end{aligned}$$

We see that the second constraint is redundant. By subtracting the two equality constraints we get

$$\begin{aligned} & -1 \le x_1 + x_2 \le 1 \\ & x_1 + x_2 - u_1 = 0 \\ & x_2 + u_0 + u_1 = 0 \\ & -1 \le u_k \le 1, \quad k = 0, 1 \end{aligned}$$

The first equality constraint is implied by the first inequality constraint since $-1 \le u_1 \le 1$. It follows that

$$\mathbf{X}_{0,feas} = \{x : -1 \le x_1 + x_2 \le 1, \ -2 \le x_2 \le 2\}.$$

In the the next problem we derive an explicit MPC in the case of a linear cost function over the shortest possible prediction horizon, i.e. $N = 1$.

Example 2. Consider a model predictive control problem where at each time instance the following optimization problem is solved: $\min_{U_t} J(x_{t|t}, U_t)$ where

$$J(x_{t|t}, U_t) = |u_{t|t}| \qquad \text{subject to} \begin{cases} x_{t+1|t} = Ax_{t|t} + Bu_{t|t} \\ -1 \le Cx_{t+1|t} \le 1, \\ -1 \le u_{t|t} \le 1, \end{cases}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Derive the explicit MPC controller.
Solution: The optimization problem becomes

$$J^*(t,x) = \min |u_{t|t}| \text{ subject to } \begin{cases} -1 \leq CAx_{t|t} + CBu_{t|t} \leq 1 \\ -1 \leq u_{t|t} \leq 1 \end{cases}$$
$$\min |u_{t|t}| \text{ subject to } \begin{cases} -1 \leq Fx_{t|t} + u_{t|t} \leq 1 \\ -1 \leq u_{t|t} \leq 1 \end{cases}$$

where $F = \begin{bmatrix} 1 & 2 \end{bmatrix}$. We see that the optimal feedback control is

$$u^*_{t|t} = \begin{cases} 1 - Fx_{t|t}, & 1 \leq Fx_{t|t} \leq 2 \\ 0, & -1 \leq Fx_{t|t} \leq 1 \\ -1 - Fx_{t|t}, & -2 \leq Fx_{t|t} \leq -1 \end{cases}$$

the problem is infeasible if $|Fx_{t|t}| > 2$