## Linear Quadratic Control

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## The LQ control theorem

The standard linear quadratic problem

$$\underbrace{\min_u x(t_1)^T S x(t_1) + \int_{t_0}^{t_1} (x(t)^T Q(t) x(t) + u(t)^T R(t) u(t)) dt}_{J(u)} \tag{1}$$

$$\text{subj. to} \quad \dot{x}(t) = A(t)x(t) + B(t)u(t), \; x(t_0) = x_0$$

**Theorem 1.** *The solution to (1) is given by the state feedback* $u(t) = -R^{-1}B^T P(t)x(t)$, *where* $P(t) = P(t)^T$ *solves the Riccati equation*

$$\dot{P} = A^T P + PA + Q - PBR^{-1}B^T P = 0, \quad P(t_1) = S \tag{2}$$

*The optimal cost is* $\min_u J(u) = x_0^T P(t_0) x_0$.



## Solving the Riccati Equation

The next result is an alternative to directly solving the nonlinear matrix differential equation (2).

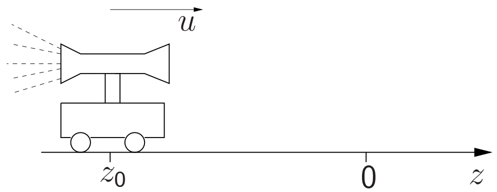
**Theorem 2.** *The solution to (2) is given as* $P(t) = Y(t)X^{-1}(t)$, *where*

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}}_{H} \begin{bmatrix} X \\ Y \end{bmatrix}, \quad \begin{bmatrix} X(t_1) \\ Y(t_1) \end{bmatrix} = \begin{bmatrix} I \\ S \end{bmatrix}$$

*The matrix* $H$ *is a so-called Hamiltonian matrix. Such matrices appear frequently in systems theory.*



## The Rocket Car




Drive the rocket car from rest at position $z_0$ to approximately rest at position 0.

$$\min \|x(T)\|^2 + \int_0^T u(t)^2 dt \quad \text{subj to} \quad \begin{cases} \dot{x} = Ax + Bu, \\ x(0) = \begin{bmatrix} z_0 \\ 0 \end{bmatrix} \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We use Theorem 2 (above) to compute the solution to the Riccati equation

$$\begin{bmatrix}\dot{X}\\ \dot{Y}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0\end{bmatrix}\begin{bmatrix}X\\ Y\end{bmatrix}, \quad \begin{bmatrix}X(T)\\ Y(T)\end{bmatrix} = \begin{bmatrix}I\\ I\end{bmatrix}$$

which gives

$$\begin{bmatrix}X(t)\\ Y(t)\end{bmatrix} = \begin{bmatrix}1+(t-T)^3/6 & t-T-(t-T)^2/2\\ (t-T)^2/2 & 1-(t-T)\\ 1 & 0\\ -(t-T) & 1\end{bmatrix}$$



$$P(t) = \begin{bmatrix}1 & 0\\ -(t-T) & 1\end{bmatrix}\begin{bmatrix}1+(t-T)^3/6 & t-T-(t-T)^2/2\\ (t-T)^2/2 & 1-(t-T)\end{bmatrix}^{-1}$$

$$= \frac{\begin{bmatrix}1+T-t & T-t+(T-t)^2/2\\ T-t+(T-t)^2/2 & 1+(T-t)^2+\frac{1}{3}(T-t)^3\end{bmatrix}}{1+T-t+(T-t)^3/3+(T-t)^4/12}$$

$$K(t) = -R^{-1}B^TP(t) = \frac{\begin{bmatrix}T-t+(T-t)^2/2 & 1+(T-t)^2+\frac{1}{3}(T-t)^3\end{bmatrix}}{1+T-t+(T-t)^3/3+(T-t)^4/12}$$