Outline

- Second order variations
  - Sufficient conditions for (local) optimality
  - Local control along trajectory
- Gradient methods
- Newtons method

Basic Ideas in NLP Generalizes to Optimal Control

Second Order Sufficient Conditions

Consider the NLP

$$\min f(x) \quad \text{subject to} \quad G(x) = 0 \tag{1}$$

Second order sufficient condition:
Suppose there exists x^* and $\lambda \in \mathbf{R}^p$ such that

(i) $G(x^*) = 0$

(ii) $\nabla_x l(x^*, \lambda) = 0$, where $l(x, \lambda) = f(x) + \lambda^T G(x)$ is the *Lagrangian.*

(iii) $L(x^*, \lambda) = f_{xx}(x^*) + \sum_{k=1}^{p} \lambda_k (g_k)_{xx}(x^*)$ is positive definite on

$$M = \{y \in \mathbf{R}^n : G_x(x^*)y = 0\}$$

Then x^* is a local minimum of (1).

This result generalizes to optimal control problems, i.e. if

(i) Constraint is satisfied (dynamic equation)

(ii) Gradient of Lagrangian is zero

(iii) Hessian is positive definite on tangent space of the constraint set ("the linearized constraint space")

then we have a (locally) optimal solution.

Second Order Variation

$$\begin{aligned} &\min \underbrace{\phi(x(t_f)) + \int_0^{t_f} f_0(t, x(t), u(t))dt}_{\mathcal{F}(u(\cdot))} \\ &\text{subj. to} \quad \underbrace{f(t, x(t), u(t)) - \dot{x}(t) = 0, \; x(0) = x_0}_{\mathcal{G}(u(\cdot))=0} \end{aligned} \tag{2}$$

Lagrangian $l(u(\cdot), \lambda(\cdot)) = \mathcal{F}(u(\cdot)) + \lambda(\cdot) \cdot \mathcal{G}(u(\cdot))$

$$l(u(\cdot), \lambda(\cdot)) = \phi(x(t_f)) + \int_0^{t_f} \big(H(t, x(t), u(t), \lambda(t)) - \lambda(t)^T \dot{x}(t)\big)\, dt$$

where $H(t, x, u, \lambda) = f_0(t, x, u) + \lambda^T f(t, x, u)$

Consider a solution $(x^*(\cdot), u^*(\cdot))$ and choose

$$\dot{\lambda}(t) = -H_x(t, x^*(t), u^*(t), \lambda(t)), \quad \lambda(t_f) = \phi_x(x^*(t_f))$$

Taylor series expansion

$$l(u(\cdot),\lambda(\cdot)) = l(u^*(\cdot),\lambda(\cdot)) + \phi_x(x^*(t_f))\delta x(t_f) +$$
$$\int_0^{t_f} H_x^T(t,x^*,u^*,\lambda)\delta x + H_u^T(t,x^*,u^*,\lambda)\delta u - \lambda^T\delta\dot{x}\,dt +$$
$$\frac{1}{2}\delta x(t_f)^T\phi_{xx}(x^*(t_f))\delta x(t_f) + \frac{1}{2}\int_0^{t_f}\begin{bmatrix}\delta x\\ \delta u\end{bmatrix}^T\begin{bmatrix}H_{xx}^* & H_{xu}^*\\ H_{ux}^* & H_{uu}^*\end{bmatrix}\begin{bmatrix}\delta x\\ \delta u\end{bmatrix}dt + h.o.t$$
$$= l(u^*(\cdot),\lambda(\cdot)) + \underbrace{(\phi_x(x(t_f)) - \lambda(t_f))^T}_{0}\delta x(t_f) + \int_0^{t_f}\underbrace{(H_x^* + \dot{\lambda})^T}_{0}\delta x\,dt$$
$$\underbrace{\int_0^{t_f}(H_u^*)^T\delta u}_{\nabla l(u^*(\cdot),\lambda(\cdot))\cdot\delta(\cdot)} + \underbrace{\frac{1}{2}\delta x(t_f)^T\phi_{xx}^*\delta x(t_f) + \frac{1}{2}\int_0^{t_f}\begin{bmatrix}\delta x\\ \delta u\end{bmatrix}^T\begin{bmatrix}H_{xx}^* & H_{xu}^*\\ H_{ux}^* & H_{uu}^*\end{bmatrix}\begin{bmatrix}\delta x\\ \delta u\end{bmatrix}dt}_{\frac{1}{2}L(u^*(\cdot),\lambda(\cdot))\cdot(\delta x(\cdot),\delta u(\cdot))} + h.o.t$$

Gradient of Lagrangian:

$$\nabla_u l(u^*(\cdot),\lambda(\cdot))\cdot\delta u(\cdot) = \int_0^{t_f} H_u(t,x^*,u^*,\lambda)^T\delta u\,dt$$

It is the gradient of the cost function "$\nabla J(u(\cdot))$" and can be stated as

$$\nabla_u l(u^*(\cdot),\lambda(\cdot)) = H_u(t,x^*(\cdot),u^*(\cdot),\lambda(\cdot))$$

Hessian:

$$L(u^*(\cdot),\lambda(\cdot))\cdot(\delta x(\cdot),\delta u(\cdot)) = \delta x(t_f)^T\phi_{xx}\delta x(t_f) +$$
$$\int_0^{t_f}\begin{bmatrix}\delta x\\ \delta u\end{bmatrix}^T\begin{bmatrix}H_{xx}^* & H_{xu}^*\\ H_{ux}^* & H_{uu}^*\end{bmatrix}\begin{bmatrix}\delta x\\ \delta u\end{bmatrix}dt$$

where $H_{xx}^* = H_{xx}(t,x^*,u^*,\lambda)$, e.t.c.
Linearized constraint $M = \{(\delta x,\delta u) : \delta\dot{x} = f_x^*\delta x + f_u^*\delta u,\ \delta x(0) = 0\}$

Positivity Condition

Proposition 1. *The following are equivalent*

(i) Hessian is positive over the linearized dynamics, i.e., $\exists\epsilon > 0$ s.t.

$$L(u^*(\cdot),\lambda(\cdot))\cdot(\delta x(\cdot),\delta u(\cdot)) \geq \epsilon(\|\delta x(\cdot)\|^2 + \|\delta u(\cdot)\|^2)$$

for all $\delta\dot{x} = f_x(t,x^*,u^*)\delta x + f_u(t,x^*,u^*)\delta u$, $\delta x(0) = 0$

(ii) $R > 0$ and there exists $P(t) = P(t)^T$ satisfying

$$\dot{P} + A^TP + PA + Q = (PB+S)R^{-1}(PB+S)^T,\ \ P(t_f) = \phi_{xx}(x^*(t_f))$$

where

$$A = f_x(t,x^*,u^*),\ \ B = f_u(t,x^*,u^*),\quad \begin{bmatrix}Q & S\\ S^T & R\end{bmatrix} = \begin{bmatrix}H_{xx}^* & H_{xu}^*\\ H_{ux}^* & H_{uu}^*\end{bmatrix}$$

Proof sketch: One can show that

$$\frac{d}{dt}\delta x^TP\delta x + \begin{bmatrix}\delta x\\ \delta u\end{bmatrix}^T\begin{bmatrix}Q & S\\ S^T & R\end{bmatrix}\begin{bmatrix}\delta x\\ \delta u\end{bmatrix} = \underbrace{(\delta u - L\delta x)^TR(\delta u - L\delta x)}_{|\delta u - L\delta x|^2_{R^{1/2}}}$$

where $L = -R^{-1}(B^TP + S)$. Integration gives

$$\delta x(t_f)^T\phi_{xx}\delta x(t_f) + \int_0^{t_f}\begin{bmatrix}\delta x\\ \delta u\end{bmatrix}^T\begin{bmatrix}Q & S\\ S^T & R\end{bmatrix}\begin{bmatrix}\delta x\\ \delta u\end{bmatrix}dt$$
$$= \int_0^{t_f}|\delta u - L\delta x|^2_{R^{1/2}}dt \geq 0$$

since $R > 0$. A more elaborate argument proves strict positivity.

Sufficient Conditions

Theorem 1. *Suppose there exists $x^*(\cdot), u^*(\cdot))$ and $\lambda(\cdot)$ such that*

(i) They satisfy the Hamiltonian system

$$\dot{x}^* = H_\lambda(t, x^*, u^*, \lambda) = f(t, x^*, u^*), \; x^*(0) = x_0$$
$$\dot{\lambda} = -H_x(t, x^*, u^*, \lambda), \quad \lambda(t_f) = \phi_x(x^*(t_f))$$

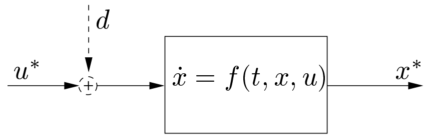
(ii) Gradient is zero (pointwise minimization) $H_u(t, x^*, u^*, \lambda) = 0$

(iii) Hessian positive definite over linearized dynamics, i.e. $R > 0$ *and exists* $P = P^T$ *such that*

$$\dot{P} + (f_x^*)^T P + P f_x^* + H_{xx}^* = (P f_u^* + H_{xu}^*)(H_{uu}^*)^{-1}(P f_u^* + H_{xu}^*)^T,$$
$$P(t_f) = \phi_{xx}(x^*(t_f))$$

Then $(x^*(\cdot), u^*(\cdot))$ *is a local minimum of (2).*

Open Loop Control



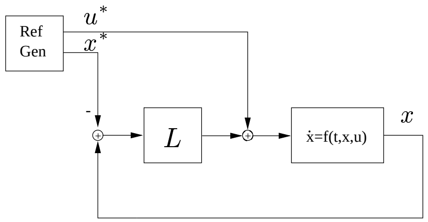
- Using PMP to design an optimal trajectory $x^*(\cdot)$

$$\min \phi(x(t_f)) + \int_0^{t_f} f_0(t, x(t), u(t)) dt \quad \text{subj. to} \quad \dot{x} = f(t, x, u)$$

gives an *open loop* control function $u^*(\cdot)$.

- Open loop controls can be sensitive to disturbances and model errors, e.g. a small disturbance d on the control may change the solution to $x^* + \Delta x$, for some big Δx

Local control around reference trajectory using LQ

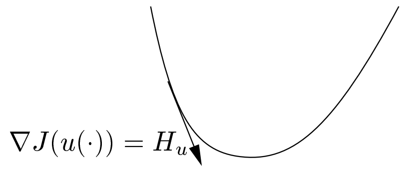


- Optimal reference trajectory computed using PMP
- Feedback controller improves robustness $u = L(x - x^*) + u^*$

$$\min \delta x(t_f)^T \phi_{xx}^* \delta x(t_f) + \int_0^{t_f} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix}^T \begin{bmatrix} H_{xx}^* & H_{xu}^* \\ H_{ux}^* & H_{uu}^* \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} dt$$

subj. to $\quad \delta \dot{x} = f_x^* \delta x + f_u^* \delta u$

Gradient Method in Optimal Control



1. Guess $u(t)$, $0 \le t \le t_f$
2. Integrate forward $\dot{x} = f(t, x, u)$, $x(0) = x_0$
3. Integrate backward $\dot{\lambda} = -H_x(t, x, u, \lambda)$, $\lambda(t_f) = \phi_x(x(t_f))$
4. $u_{new}(t) := u(t) - \alpha H_u(t, x(t), u(t), \lambda(t))$

Repeat Step 2-4 until $\int_0^{t_f} |H_u(t, x(t), u(t), \lambda(t))|^2 dt < \epsilon$

(+) It gives good improvement in the first iterations

(+) Stability is generally improved compared to the shooting method since integration of $x(\cdot)$ and $\lambda(\cdot)$ are performed in stable direction.

(+) Control constraints can easily be taken into account

(+) Used to solve large number of aeronautical problems in the 1960s.

(−) Convergence tends to be slow.

Newtons Method

- Iterative technique for solving $G(x) = 0$
- Consider the Taylor series expansion

$$0 = G(x + \delta x) = G(x) + G_x(x)\delta x + h.o.t$$

- Solve the linearized equation recursively

$$G(x) + G_x(x)\delta x = 0$$

by updating x as

$$x_{new} := x - G_x(x)^{-1}G(x)$$

Newtons Method in NLP

$$\min f(x) \quad \text{subject to} \quad G(x) = 0$$

First order necessary conditions: Exists λ s.t.

$$\begin{cases} \nabla l(x^*, \lambda) = 0 \\ G(x^*) = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} f_x(x^*) + G_x(x^*)^T\lambda = 0 \\ G(x^*) = 0 \end{cases}$$

Linearized first order equations (around (x^*, λ))

$$\begin{aligned} \nabla l(x^*, \lambda) + L(x^*, \lambda)\delta x + G_x(x^*)^T\delta\lambda &= 0 \\ G(x^*) + G_x(x^*)^T\delta x &= 0 \end{aligned} \tag{3}$$

This corresponds to the first order conditions of QP

$$\min \nabla l(x^*, \lambda)\delta x + \frac{1}{2}\delta x^T L(x^*, \lambda)\delta x \quad \text{subj. to} \quad G(x^*) + G_x(x^*)\delta x = 0$$

Solve QP recursively

Step 1 Guess initial values (x, λ)

Step 2 Solve the QP

$$\begin{aligned} \min \nabla l(x^*, \lambda)\delta x + \frac{1}{2}\delta x^T L(x^*, \lambda)\delta x \\ \text{subject to} \quad G(x^*) + G_x(x^*)\delta x = 0 \end{aligned}$$

Step 3 Update

$$\begin{aligned} x_{new} &:= x + \delta x \\ \lambda_{new} &:= \lambda + \delta\lambda \end{aligned}$$

Step 4 Stop if $|\nabla l(x, \lambda)|^2 + |G(x)|^2 \leq \epsilon$, otherwise go to step 2.

Newtons Method in Optimal Control

Solve recursively the LQ problem

$$\begin{aligned} &\min \nabla l(u(\cdot),\lambda(\cdot))\cdot\delta u(\cdot) + \frac{1}{2}L(u(\cdot),\lambda(\cdot))\cdot(\delta x(\cdot),\delta u(\cdot)) \\ &\text{subject to}\quad \delta\dot{x} = f_x\delta x + f_u\delta u \end{aligned} \tag{4}$$

or equivalently

$$\min \frac{1}{2}\delta x(t_f)^T\phi_{xx}\delta x(t_f) + \int_0^{t_f}\left(H_u\delta u + \frac{1}{2}\begin{bmatrix}\delta x\\ \delta u\end{bmatrix}^T\begin{bmatrix}H_{xx} & H_{xu}\\ H_{xu}^T & H_{uu}\end{bmatrix}\begin{bmatrix}\delta x\\ \delta u\end{bmatrix}\right)dt$$

subject to $\delta\dot{x} = f_x\delta x + f_u\delta u$

This LQ problem can be solved using DynP in a standard way. We get

$$\delta u = -R^{-1}\left((B^TP + S^T)\delta x + r + B^Tq\right)$$

where

$$\begin{aligned} &\dot{P} + PA + A^TP + Q = (PB+S)R^{-1}(PB+S)^T,\ P(t_f) = \phi_{xx}(x(t_f)) \\ &\dot{q} + A^Tq = (PB+S)R^{-1}(B^Tq + r),\quad q(t_f) = 0 \end{aligned} \tag{5}$$

where

$$A = f_x,\ B = f_u,\ r = H_u,\ \begin{bmatrix}Q & S\\ S^T & R\end{bmatrix} = \begin{bmatrix}H_{xx} & H_{xu}\\ H_{ux} & H_{uu}\end{bmatrix}$$

1. Guess $u(t)$, $0 \le t \le t_f$
2. Integrate $\dot{x}(t) = f(t, x(t), u(t))$, $x(0) = x_0$ forward direction
3. Integrate $\dot{\lambda}(t) = -H_x(t, x(t), u(t), \lambda(t))$, $\lambda(t_f) = \phi_x(x(t_f))$ in the backward direction.
4. Solve for $P(\cdot)$ and $q(\cdot)$ in (5)
5. Update, using the feedback law

   $$u_{new} := u - \alpha R^{-1}[(B^TP + S^T)(x_{new} - x) + (r + B^Tq)]$$

   where $\alpha \in (0,1)$ is a step length parameter (should be equal to one in a proper Newton method).

Repeat steps 2 to 5 until $\|H_u(t, x(t), u(t), \lambda(t))\| \le \epsilon$

(+) Fast convergence

(+) Solid theoretical justification

(−) Good initial guess is needed. Can be achieved by using a gradient method initially.

(−) Each iteration is computationally quite expensive.