Exercise 16.1

We have $f'(x) = 2x - \cos x$ and $f''(x) = 2 + \sin x$.
Since $|\sin x| \leq 1$, $f''(x) > 0$ for all x.
If $x^{(k)}$, $k = 0, 1, 2, 3, \ldots$ denote the iterates in Newton's method, then we have for $k \geq 1$ that

$$x^{(k+1)} = x^{(k)} - \frac{f'(x^{(k)})}{f''(x^{(k)})}$$

$$= x^{(k)} - \frac{2x^{(k)} - \cos x^{(k)}}{2 + \sin x^{(k)}}$$

So if we start from $x^{(0)} = 1$, then we obtain

$$x^{(1)} = 0.4863,$$
$$x^{(2)} = 0.4504,$$
$$x^{(3)} = 0.4502.$$

Hence $\hat{x} = 0.45$ (to two decimal places).

Exercise 16.2

We have $\nabla f(x) = \begin{bmatrix} 4x_1^3 + 4x_1x_2^2 & 4x_1^2x_2 + 4x_2^3 \end{bmatrix}$

and $F(x) = \begin{bmatrix} 12x_1^2 + 4x_2^2 & 8x_1x_2 \\ 8x_1x_2 & 4x_1^2 + 12x_2^2 \end{bmatrix}$.

The increment d is obtained by solving

$$F(x^{(k)})\, d = -\nabla f(x^{(k)})$$

and so if the current iterate is (a, a) then we have

$$\begin{bmatrix} 16a^2 & 8a^2 \\ 8a^2 & 16a^2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} -8a^3 \\ -8a^3 \end{bmatrix}$$

that is,

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} -a \\ -a \end{bmatrix}$$

and so $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -a \\ -a \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -a \\ -a \end{bmatrix}$.

Thus

$$x^{(k+1)} = x^{(k)} + d = \begin{bmatrix} a \\ a \end{bmatrix} + \begin{bmatrix} -\frac{a}{3} \\ -\frac{a}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}a \\ \frac{2}{3}a \end{bmatrix},$$

So if we start from $x^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

then $x^{(k)} = \begin{bmatrix} \left(\frac{2}{3}\right)^k \\ \left(\frac{2}{3}\right)^k \end{bmatrix}$. As $\left(\frac{2}{3}\right)^k \longrightarrow 0$ when $k \to \infty$,

we see that $\hat{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(Note that $f(x_1, x_2) = (x_1^2 + x_2^2)^2$ which clearly has a global minimum at $(0, 0)$.)

Exercise 16.3.

(1) We have $\nabla f(x) = \left[4x_1^3 - 3x_1^2 + 2x_1 - 1 \quad \cdots \quad 4x_n^3 - 3x_n^2 + 2x_n - 1\right]$.

So the Hessian
$$F(x) = \begin{bmatrix} 12x_1^2 - 6x_1 + 2 & & \\ & \ddots & \\ & & 12x_n^2 - 6x_n + 2 \end{bmatrix}$$

But
$$\begin{aligned} 12x_j^2 - 6x_j + 2 &= 3(4x_j^2 - 2x_j + 2) \\ &= 3\left((2x_j)^2 - 2.(2x_j)\left(\tfrac{1}{2}\right) + \tfrac{1}{4} + 2 - \tfrac{1}{4}\right) \\ &= 3\left(\left(2x_j - \tfrac{1}{2}\right)^2 + \tfrac{7}{4}\right) > 0 \quad \forall\, x_j. \end{aligned}$$

Hence $F(x)$ is positive definite $\forall x$.

So f is convex.

(2) The Newton direction $d^{(k)}$ is determined via the system
$$F(x^{(k)})\, d^{(k)} = -\nabla f(x^{(k)})^T$$

In ourcase $x^{(1)} = [1 \cdots 1]^T$,
$$\nabla f(x^{(1)}) = [2 \ \cdots \ 2]$$
$$F(x^{(k)}) = 8I.$$

Hence
$$d^{(1)} = -\frac{1}{8}\begin{bmatrix} 2 \\ \vdots \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ \vdots \\ -\frac{1}{4} \end{bmatrix}.$$

So
$$x^{(2)} = x^{(1)} + d^{(1)} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{4} \\ \vdots \\ -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ \vdots \\ \frac{3}{4} \end{bmatrix}.$$