Exercise 21.3

Let $x, y \in \mathcal{K}$ and $t \in (0,1)$. Then

$$\left.\begin{array}{l} g_i(x) \leq 0 \\ g_i(y) \leq 0 \end{array}\right\} \quad i = 1, \dots, m.$$

Since each g_i is convex, we also have

$$g_i(tx + (1-t)y) \leq \underbrace{t}_{>0} \underbrace{g_i(x)}_{\leq 0} + \underbrace{(1-t)}_{>0} \underbrace{g_i(y)}_{\leq 0}$$

$$\leq 0.$$

Thus $tx + (1-t)y \in \mathcal{K}$ as well. Hence $\mathcal{K}$ is convex.

Exercise 21.9

(1) S1 $\underline{0} \in \mathcal{S}$ since $M_i \underline{0} = \underline{0}$ for $i = 1, \dots, k$.

S2 If $x, y \in \mathcal{S}$, then for $i = 1, \dots, k$,
$$M_i x = x \quad \text{and} \quad M_i y = y.$$
But then $M_i(x+y) = M_i x + M_i y = x + y$ for all $i \in \{1, \dots, k\}$.
Hence $x + y \in \mathcal{S}$.

S3 If $x \in \mathcal{S}$ and $\alpha \in \mathbb{R}$, then $M_i(\alpha x) = \alpha M_i x = \alpha x$ for $i = 1, \dots, k$, and so $\alpha x \in \mathcal{S}$.

Consequently $\mathcal{S}$ is a subspace of $\mathbb{R}^n$.

(2) Let $x \in \mathbb{R}^n$. We have $\bar{x} = \frac{1}{k} \sum_{i=1}^{k} M_i x$.
Let $i_* \in \{1, \dots, k\}$.
Then for each $j \in \{1, \dots, k\}$, there exists a unique $i \in \{1, \dots, k\}$ such that $M_j = M_{i_*} M_i$. Indeed, $M_i = M_{i_*}^{-1} M_j$.

Hence $M_{i_*} \bar{x} = M_{i_*} \left(\frac{1}{k} \sum_{i=1}^{k} M_i x \right) = \frac{1}{k} \sum_{i=1}^{k} M_{i_*} M_i x = \frac{1}{k} \sum_{j=1}^{k} M_j x$
$= \bar{x}$.

(3) Let $x \in \mathbb{R}^n$. Since f is G-invariant, we have $f(M_i x) = f(x)$ for all $i \in \{1, \dots, k\}$. By the convexity of f, it follows that
$$f(\bar{x}) = f\left(\frac{1}{k} \sum_{i=1}^{k} M_i x \right) \le \frac{1}{k} \sum_{i=1}^{k} f(M_i x) = \frac{1}{k} \sum_{i=1}^{k} f(x) = \frac{1}{k} k f(x) = f(x).$$

(4) Let $x_* \in \mathcal{F}$ be a global optimal solution to (P).
Let $x \in \mathcal{F}$. We have $f(\bar{x}_*) \le f(x_*) \le f(x)$. Also $x_* \in \mathcal{F}$ and so $M_j x_* \in \mathcal{F}$ (since $\mathcal{F}$ is G-invariant). By the convexity of $\mathcal{F}$, also $\bar{x}_* = \frac{1}{k} \sum_{j=1}^{k} M_j x_* \in \mathcal{F}$. So we have that $\bar{x}_* \in \mathcal{F}$ and for all $x \in \mathcal{F}$, $f(\bar{x}_*) \le f(x)$.
This means that $\bar{x}_*$ is a global optimal solution to (P).
But $\bar{x}_* \in \mathcal{S}$, and this proves the claim.

(5) If $x_* = \begin{bmatrix} x_{1*} \\ \vdots \\ x_{n*} \end{bmatrix}$ is a global optimal solution,

then $\overline{x}_* = \frac{1}{n!} \sum_{\pi} P_{\pi} x_* = \frac{1}{n!} \begin{bmatrix} \sum_{\pi} x_{\pi(1)*} \\ \vdots \\ \sum_{\pi} x_{\pi(n)*} \end{bmatrix}$.

Let $i \neq j$.

Then

$$\sum_{\pi} x_{\pi(i)*} = \sum_{\pi} x_{\pi(\tau(j))*}, \quad \text{where } \tau \text{ is the transposition } i \leftrightarrow j$$

$$= \sum_{\pi} x_{(\pi \circ \tau)(j)*}$$

$$= \sum_{\sigma} x_{\sigma(j)*}.$$

So we see that all the entries of $\overline{x}_*$ are equal.

This proves the claim.

(6) By the previous results*, we seek an optimal solution (x, y, z) of the form (a, a, a) which satisfies the constraints:

$$3a^2 \leq 1 \quad \text{and so} \quad a^2 \leq \frac{1}{3};$$

$$3a \leq 1 \quad \text{and so} \quad a \leq \frac{1}{3}.$$

So we arrive at the following problem:

$$\begin{cases} \text{maximize} & 3a^4 \\ \text{subject to} & a^2 \leq \frac{1}{3}, \\ & a \leq \frac{1}{3} \end{cases}$$

Clearly the solution is $a = \frac{-1}{\sqrt{3}}$, that is, $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ is optimal for the original problem.

(* Since the feasible set is compact, by Theorem 1.16, we know that there is a global optimal solution.)

Exercise 21.10

(1) No.

For example with $x = (2, 0, 0)$, $y = (0, 2, 2)$ we have $f(x) = 0 = f(y)$, and $f(\frac{x+y}{2}) = f(1,1,1) = 1$.

If the function was convex, we would obtain

$$f\left(\frac{x+y}{2}\right) = 1 \leq 0 = \frac{f(x)+f(y)}{2},$$

a contradiction. Hence f is not convex.

(2) $f(x) = (x_1 x_2^2 x_3^3)^2 \geqslant 0 = f(\hat{x}) \quad \forall x \in \mathbb{R}^3$ and so $\hat{x} = 0$ is a global minimizer.

(3) Since $(-x_1)^2 = x_1^2$, $(-x_2)^4 = x_2^4$ and $(-x_3)^6 = x_3^6$, we can assume that $x_1 \geqslant 0$, $x_2 \geqslant 0$, $x_3 \geqslant 0$ without loss of generality. Moreover, if any of x_1, x_2 or x_3 are zero, then $f(x_1, x_2, x_3) = 0$, which is clearly not the maximum value of f on the unit sphere (for example $f(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) > 0$.). So we may assume that in fact $x_1 > 0$, $x_2 > 0$ and $x_3 > 0$.

The problem is equivalent to:

$$\begin{array}{ll} \text{maximize} & \log f(x) = 2\log x_1 + 4\log x_2 + 6\log x_3 \\ \text{subject to} & x_1^2 + x_2^2 + x_3^2 \leq 1 \\ & x_1, x_2, x_3 > 0 \end{array}$$

(Reason: $e^{\cdot}$ and $\log \cdot$ are strictly increasing functions).

But now this is further equivalent to:

$$(*)\left\{\begin{array}{ll} \text{minimize} & -2\log x_1 - 4\log x_2 - 6\log x_3 \\ \text{subject to} & x_1^2 + x_2^2 + x_3^2 \leq 1, \\ & x_1, x_2, x_3 > 0. \end{array}\right.$$

This is a <u>convex</u> optimization problem.

We use the KKT algorithm to solve it.

We have

(KKT-1) $\nabla f(x) + y^T \nabla g(x) = 0$

i.e, $\begin{bmatrix} -\frac{2}{x_1} & -\frac{4}{x_2} & -\frac{6}{x_3} \end{bmatrix} + y \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 \end{bmatrix} = 0$

i.e, $y = \frac{1}{x_1^2} = \frac{2}{x_2^2} = \frac{3}{x_3^2}$

(KKT-2) $g(x) \leq 0$

i.e, $x_1^2 + x_2^2 + x_3^2 \leq 1$

(KKT-3) $y \geq 0$

But from (KKT-1) we have $y = \frac{1}{x_1^2} > 0$.

(KKT-4) $y^T g(x) = 0$

Since $y > 0$, we have $g(x) = 0$.

So $x_1^2 + x_2^2 + x_3^2 = 1$.

Using (KKT-1), this gives $\frac{1}{y} + \frac{2}{y} + \frac{3}{y} = 1$

i.e, $y = 6$.

So $x_1 = \frac{1}{\sqrt{6}}$, $x_2 = \frac{1}{\sqrt{3}}$, $x_3 = \frac{1}{\sqrt{2}}$.

So the only global optimal solution to (*) is $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}\right)$.

Consequently, our original problem has 8 solutions:

$\left(\pm\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{2}}\right)$.

Exercise 21.11

(1) f is convex.

Let
$$g_1(x) = x_1 - 2$$
$$g_2(x) = x_2 - 2$$
$$g_3(x) = x_3 - 2$$
$$g_4(x) = -x_1$$
$$g_5(x) = -x_2$$
$$g_6(x) = -x_3.$$

Then the problem becomes:

minimize $f(x)$

subject to $g_i(x) \leq 0$, $i = 1, \dots, 6$.

Since f, g_i $(i = 1, \dots, 6)$ are convex, this is a convex problem.

We use the KKT-algorithm to check if $\hat{x} = (2, 1, 0)$ is a global optimal solution.

(KKT-2) holds since $\hat{x}$ is clearly feasible.

(KKT-4) $\hat{y}^T g(\hat{x}) = 0$

But $g_1(\hat{x}) = g_6(\hat{x}) = 0$, and $g_2(\hat{x}), g_3(\hat{x}), g_4(\hat{x}), g_5(\hat{x}) \neq 0$

So $\hat{y}_2 = \hat{y}_3 = \hat{y}_4 = \hat{y}_5 = 0$ So (KKT-2) holds for $\hat{y}_2, \hat{y}_3, \hat{y}_4, \hat{y}_5$.

(KKT-1) $\nabla f(\hat{x}) + \hat{y}^T \nabla g(\hat{x}) = 0$

So $2(\hat{x}_1 + \hat{x}_2) + 2(\hat{x}_1 + \hat{x}_3) - 12 + \hat{y}_1 - \underbrace{\hat{y}_4}_{=0} = 0$

$$\hat{y}_1 = -\left[2(2+1) + 2(2+0) - 12\right]$$
$$= -[6 + 4 - 12]$$
$$= 2 \geq 0$$

Hence (KKT-3) holds for $\hat{y}_1$

Also, $2(\hat{x}_1 + \hat{x}_2) + 2(\hat{x}_2 + \hat{x}_3) - 8 + \underbrace{\hat{y}_2}_{=0} - \underbrace{\hat{y}_5}_{=0} = 0$.

i.e., $2(2+1) + 2(1+0) - 8 = 6 + 2 - 8 = 0$,

which is indeed true.

Finally, $2(\hat{x}_2 + \hat{x}_3) + 2(\hat{x}_3 + \hat{x}_1) - 4 + \underbrace{\hat{y}_3}_{=0} - \hat{y}_6 = 0$

i.e., $\hat{y}_6 = 2(1+0) + 2(0+2) - 4$
$$= 2 + 4 - 4 = 2 \geq 0.$$

Hence (KKT-3) holds for $\hat{y}_6$

So all the (KKT) conditions are satisfied.

Hence $\hat{x}$ is a globally optimal solution.

(2) Let $g(x) := (x_1 - c_1)^2 + (x_2 - c_2)^2 + (x_3 - c_3)^2 - 1$. g is convex.

So
$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & g(x) \leq 0 \end{cases}$$
is a convex problem.

We use the KKT-algorithm.

(KKT-1) $\nabla f(\hat{x}) + \hat{y}^T \nabla g(\hat{x}) = 0$

$$\begin{cases} 2(2+1) + 2(0+2) - 12 + y \cdot 2(2 - c_1) = 0 \\ 2(2+1) + 2(1+0) - 8 + y \cdot 2(1 - c_2) = 0 \\ 2(1+0) + 2(0+2) - 4 + y \cdot 2(0 - c_3) = 0. \end{cases}$$

i.e.,
$$\begin{cases} y(2 - c_1) = 1 \\ y(1 - c_2) = 0 \\ y \, c_3 = +1 \end{cases}$$

(KKT-2) $g(\hat{x}) \leq 0$

$$(2 - c_1)^2 + (1 - c_2)^2 + c_3^2 \leq 1.$$

(KKT-3) $y \geq 0$.

But from (KKT-1) $y \neq 0$ since $y(2 - c_1) = 1$.

So $y > 0$.

Hence (KKT-1) gives $y(1 - c_2) = 0$ i.e., $1 - c_2 = 0$

Thus $c_2 = 1$.

(KKT-4) $y^T g(\hat{x}) = 0$

Since $y \neq 0$, $g(\hat{x}) = 0$

i.e., $(2 - c_1)^2 + \underbrace{(1 - c_2)^2}_{=0} + c_3^2 = 1$

i.e., $(2 - c_1)^2 + c_3^2 = 1$.

$$\left(\frac{1}{y}\right)^2 + \left(\frac{1}{y}\right)^2 = 1$$

So $\frac{2}{y^2} = 1$ and so $y = \pm\sqrt{2}$.

Since $y > 0$, $y = \sqrt{2}$.

Hence $c_1 = 2 - \frac{1}{\sqrt{2}}$ and $c_3 = \frac{1}{\sqrt{2}}$.

So $c_1 = 2 - \frac{1}{\sqrt{2}}$, $c_2 = 1$ and $c_3 = \frac{1}{\sqrt{2}}$.

Exercise 21.12.

Let $g_1(x) := 4 - x_1 - x_2$

$g_2(x) := 4 - x_2 - x_3$

$g_3(x) := 4 - x_3 - x_1$.

These are convex and $g_1(0) = g_2(0) = g_3(0) = -4 < 0$.
Thus the problem is convex and regular.
So the KKT-conditions are necessary and sufficient for global optimality.

(1) (KKT-1) $\begin{cases} x_1 - 1 - y_1 - y_2 = 0 \\ x_2 - 1 - y_1 - y_3 = 0 \\ x_3 + c - y_2 - y_3 = 0. \end{cases}$ i.e., $\begin{cases} y_1 + y_2 = 1 \\ y_1 + y_3 = 1 \\ y_2 + y_3 = 2 + c. \end{cases}$

So $y_2 = y_3 = 1 + \frac{c}{2}$ and $y_1 = -\frac{c}{2}$

(KKT-2) $(2,2,2)$ is feasible.

(KKT-3) $y \geq 0. \Leftrightarrow \left[\begin{array}{ll} c \leq -2 & \text{and} \quad c \leq 0 \\ (\because y_2 = y_3 \geq 0) & (\because y_1 \geq 0) \end{array} \right]$

So (KKT-3) holds iff $c \in [-2, 0]$.

(KKT-4) $y^T g(x) = 0$.
which is satisfied since $g_1(2,2,2) = g_2(2,2,2) = g_3(2,2,2) = 0$.

So the KKT-conditions are satisfied iff $c \in [-2, 0]$.

(2) Now $g_1(\hat{x}) = 4 - 2 - 2 = 0$, but $g_2(\hat{x}) = g_3(\hat{x}) = -2$.
Hence from (KKT-4), $y_2 = y_3 = 0$.
So (KKT-1) becomes

$y_1 = -1 + x_1 = +1$,

$y_1 = -1 + x_2 = 1$,

$x_3 + c = 0$ i.e., $4 + c = 0$

i.e., $c = -4$.

So (KKT-1) holds iff $c = -4$.

(KKT-2) holds since $(2,2,4)$ is feasible.

(KKT-3) holds since $y_1 = 1 > 0$,
$y_2 = 0 = y_3$.

Hence the KKT-conditions hold iff $c = -4$.

Exercise 21.13.

The problem is regular (since with $x = (100, 1, 1, 1)$ we have $Ax - b = \begin{bmatrix} 200 - \cancel{2} + \cancel{1} + \cancel{1} \\ 100 + 1 + \cancel{2} - \cancel{2} \end{bmatrix} - \begin{bmatrix} 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 200 \\ 101 \end{bmatrix} - \begin{bmatrix} 20 \\ 30 \end{bmatrix} > 0$

and $x > 0$) and convex. So the KKT-conditions are necessary and sufficient for global optimality.

(1) (KKT-1) says $\hat{x}^T + \hat{y}^T(-A) + \hat{z}^T(-I) = 0$.

But (KKT-4) says that $\hat{z} = 0$.

So $\hat{x}^T + \hat{y}^T(-A) = 0$ i.e., $A^T\hat{y} = \hat{x}$.

Hence $AA^T\hat{y} = A\hat{x} = b$.

But $AA^T = \begin{bmatrix} 2 & -2 & 1 & 1 \\ 1 & 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -2 & 1 \\ 1 & 2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$,

and so $\hat{y} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}^{-1} b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

So (KKT-3) is satisfied.

However, $\hat{x} = A^T\hat{y} = \begin{bmatrix} 2 & 1 \\ -2 & 1 \\ 1 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 8 \\ -4 \end{bmatrix}$,

and so $\hat{x}$ is not feasible ($\hat{x}_2 = -1 < 0$!).

So there is no global optimal solution $\hat{x}$ with $A\hat{x} = b$ and $\hat{x} > 0$.

(2) If $\hat{x}_3 = \hat{x}_4 = 0$, then $A\hat{x} = b$ has a unique solution:

$\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \end{bmatrix}$ i.e., $\hat{x}_1 = 20$, $\hat{x}_2 = 10$.

Then $\hat{x}$ is feasible, and so (KKT-2) is satisfied.

Since $\hat{x}_1, \hat{x}_2 > 0$, (KKT-4) gives $\hat{z}_1 = \hat{z}_2 = 0$.

But now (KKT-1) gives

$0 = \hat{x} - A^T\hat{y} - \hat{z} = \begin{bmatrix} 20 \\ 10 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ -2 & 1 \\ 1 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \hat{z}_3 \\ \hat{z}_4 \end{bmatrix}$. In particular, $\begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$

implying $\hat{y}_2 = 15$, $\hat{y}_1 = 5/2$.

Then $\begin{bmatrix} \hat{z}_3 \\ \hat{z}_4 \end{bmatrix} = -\begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 5/2 \\ 15 \end{bmatrix} = -\begin{bmatrix} 5/2 + 30 \\ * \end{bmatrix}$ and $\neg[\hat{z}_3 \geq 0]$, violating (KKT-3). So $\hat{x}$ is not a global optimal solution in this case either.

Exercise 21.14.

① Let the three given points be denoted by (p_i, q_i), $i = 1, 2, 3$ and let (x, y) be the sought coordinates of P. Then the problem can be formulated as:

$$\text{minimize} \quad f(x,y) := \sum_{i=1}^{3} \sqrt{(x-p_i)^2 + (y-q_i)^2}$$
$$\text{subject to} \quad (x,y) \in \mathbb{R}^2$$

The problem is convex, since the Hessian of $(x,y) \mapsto \sqrt{(x-a)^2+(y-b)^2}$ is

$$\underbrace{\frac{1}{((x-a)^2+(y-b)^2)^{3/2}}}_{\geq 0} \underbrace{\begin{bmatrix} (y-b)^2 & -(x-a)(y-b) \\ -(x-a)(y-b) & (x-a)^2 \end{bmatrix}}_{= \begin{bmatrix} y-b \\ -(x-a) \end{bmatrix} \begin{bmatrix} y-b & -(x-a) \end{bmatrix}} \succeq 0 \quad \forall (x,y) \in \mathbb{R}^2$$

So the first order condition $\nabla f(\hat{x}, \hat{y}) = 0$ is necessary and sufficient for the global optimality of $(\hat{x}, \hat{y})$. Suppose that $(\hat{x}, \hat{y}) \neq (p_i, q_i)$ for $i = 1, 2, 3$.

$$\frac{\partial f}{\partial x}(\hat{x}, \hat{y}) = \sum_{i=1}^{3} \frac{\hat{x} - p_i}{\sqrt{(\hat{x}-p_i)^2 + (\hat{y}-q_i)^2}}$$

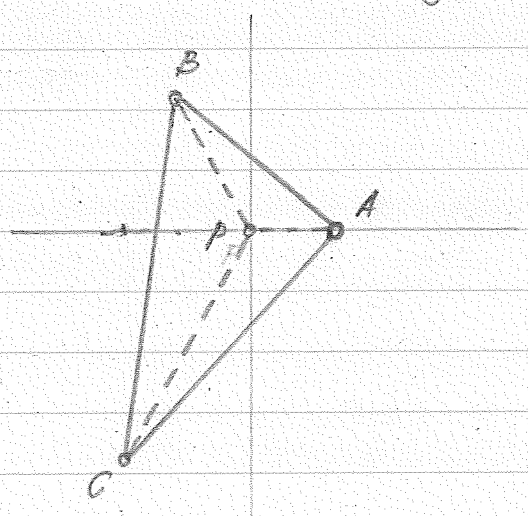
$$\frac{\partial f}{\partial y}(\hat{x}, \hat{y}) = \sum_{i=1}^{3} \frac{\hat{y} - q_i}{\sqrt{(\hat{x}-p_i)^2 + (\hat{y}-q_i)^2}}.$$

In our case, $(0,0) \notin \{A, B, C\}$, and moreover,

$$\frac{\partial f}{\partial x}(0,0) = \frac{0-1}{1} + \frac{0-(-1)}{2} + \frac{0-(-\sqrt{3})}{2\sqrt{3}} = 0, \text{ and}$$

$$\frac{\partial f}{\partial y}(0,0) = \frac{0-0}{1} + \frac{0-\sqrt{3}}{2} + \frac{0-(-3)}{2\sqrt{3}} = 0.$$

So $(0,0)$ is a global minimizer.



$\angle APC = \angle BPC = \angle CPA = 120°$.

(2) The problem can be formulated in terms of three variables $(x,y) \in \mathbb{R}^2$ and $z \in \mathbb{R}$ as follows:

$$(P)\ \begin{cases} \text{minimize} & z \\ \text{subject to} & z \geq (x-p_i)^2 + (y-q_i)^2, \quad i = 1,2,3 \end{cases}$$

(Indeed, one can check that if $(\hat{x},\hat{y}), \hat{z}$ is a solution to (P), then (1) $\hat{z} = \max\limits_i (\hat{x}-p_i)^2 + (\hat{y}-q_i)^2$, and
(2) $\forall (x,y) \in \mathbb{R}^2$, $\max\limits_i (x-p_i)^2 + (y-q_i)^2 \geq \hat{z}$.)

In our case, (P) is:

$$(P)\ \begin{cases} \text{minimize} & z \\ \text{subject to} & -z + (x-1)^2 + y^2 \leq 0, \\ & -z + (x+1)^2 + (y-\sqrt{3})^2 \leq 0, \\ & -z + (x+\sqrt{3})^2 + (y+3)^2 \leq 0. \end{cases}$$

The problem is regular (take $x = y = 0$, $z = 100$) and convex, and so the KKT-conditions are necessary and sufficient for global optimality.

(KKT-1) gives:
$$\begin{cases} \lambda_1 \cdot 2(x-1) + \lambda_2 \cdot 2(x+1) + \lambda_3 \cdot 2(x+\sqrt{3}) = 0 \\ \lambda_1 \cdot 2y + \lambda_2 \cdot 2(y-\sqrt{3}) + \lambda_3 \cdot 2(y+3) = 0 \\ 1 - \lambda_1 - \lambda_2 - \lambda_3 = 0. \end{cases}$$

i.e.,
$$\begin{cases} -\lambda_1 + \lambda_2 + \sqrt{3}\,\lambda_3 = 0 \\ -\lambda_2 + \sqrt{3}\,\lambda_3 = 0 \\ 1 - \lambda_1 - \lambda_2 - \lambda_3 = 0 \end{cases}$$

So $\lambda_3 = \dfrac{1}{3\sqrt{3}+1}$, $\lambda_2 = \dfrac{\sqrt{3}}{3\sqrt{3}+1}$, $\lambda_1 = \dfrac{2\sqrt{3}}{3\sqrt{3}+1}$.

(KKT-2) gives:
$$\begin{cases} -z + 1 \leq 0 \\ -z + 1 + 3 \leq 0 \\ -z + 3 + 9 \leq 0 \end{cases} \quad \text{i.e.} \quad \begin{cases} z \geq 1 \\ z \geq 4 \\ z \geq 12 \end{cases} \quad \text{i.e., } z \geq 12$$

(KKT-3) holds since $\lambda_1, \lambda_2, \lambda_3 \geq 0$.

(KKT-4) Since $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_3 \neq 0$, we have:
$$\begin{cases} -z + 1 = 0 \\ -z + 1 + 3 = 0 \\ -z + 3 + 9 = 0 \end{cases}$$

but this is clearly impossible.

So $(0,0)$ is not optimal.

Exercise 21.15

(1) Let x be feasible for (NLP). Then $g(x) \leq 0$.

Since g is convex, we obtain for $k = 1, \ldots, K$:

$$(0 \geq)\, g(x) \geq g(x^k) + \nabla g(x^k)(x - x^k)$$

and so $$-\nabla g(x^k)\, x \geq g(x^k) - \nabla g(x^k)\, x^k. \qquad (*)$$

Let $z := f(x)$. Since f is convex, for $k = 1, \ldots, K$ we have

$$(z =)\, f(x) \geq f(x^k) + \nabla f(x^k)(x - x^k)$$

i.e., $$z - \nabla f(x^k)\, x \geq f(x^k) - \nabla f(x^k)\, x^k, \quad k = 1, \ldots, K \qquad (**)$$

$(*)$, $(**)$ show that (x, z) is feasible for (LP). By the optimality of $(\hat{x}, \hat{z})$ for (LP), we conclude that

$$(f(x) =)\, z \geq \hat{z}.$$

(2) Since $(\hat{x}, \hat{z})$ is feasible,

$$-\nabla g(x^k)\, \hat{x} = -\nabla g(x^k)\, x^k \geq g(x^k) - \nabla g(x^k)\, x^k$$

i.e., $0 \geq g(x^k)$.

So x^k is feasible for (NLP).

Also, from part (1), $\quad f(x^k) \geq \hat{z}. \qquad (a)$

Again since $(x^k, \hat{z})$ is feasible,

$$\hat{z} - \nabla f(x^k)\, x^k \geq f(x^k) - \nabla f(x^k)\, x^k$$

i.e., $\hat{z} \geq f(x^k) \qquad (b).$

From (a) and (b), $f(x^k) = \hat{z}$. (c)

From part (1),

$$f(x) \geq \hat{z} = f(x^k) \quad \forall\, x \text{ in the feasible set of (NLP)}$$

Hence x^k is an optimal solution to (NLP).

(c) shows that the optimal values to (LP) and (NLP) are the same.

Exercise 21.16

We should make the following replacements:

$$f(x) \rightsquigarrow f(\hat{x}) + \nabla f(\hat{x})(x-\hat{x})$$

$$g_i(x) \rightsquigarrow g_i(\hat{x}) + \nabla g_i(\hat{x})(x-\hat{x}), \quad i=1,\dots,m.$$

The optimization problem is then replaced by:

$$\begin{aligned} &\text{minimize} && f(\hat{x}) + \nabla f(\hat{x})(x-\hat{x}) \\ &\text{subject to} && g_i(\hat{x}) + \nabla g_i(\hat{x})(x-\hat{x}) \leq 0, \quad i=1,\dots,m \end{aligned}$$

and removing the constant terms $f(\hat{x}), -\nabla f(\hat{x})\hat{x}$ from the cost, we obtain the linear programming problem

$$(LP) \begin{cases} \text{minimize} & \nabla f(\hat{x})\, x \\ \text{subject to} & \nabla g_i(\hat{x})\, x \leq \nabla g_i(\hat{x})\,\hat{x} - g_i(\hat{x}), \quad i=1,\dots,m \end{cases}$$

i.e.,

$$(LP) \begin{cases} \text{minimize} & c^T x \\ \text{subject to} & Ax \geqslant b \end{cases}$$

where

$$c = \nabla f(\hat{x})^T,$$

$$A = -\begin{bmatrix} \nabla g_1(\hat{x}) \\ \vdots \\ \nabla g_m(\hat{x}) \end{bmatrix},$$

$$b = A\hat{x} + g(\hat{x}).$$

The dual to (LP) is (LD), given by:

$$(LD) \begin{cases} \text{maximize} & b^T y \\ \text{subject to} & A^T y = c \\ & y \geqslant 0. \end{cases}$$

We know that:

$$(*) \begin{cases} \hat{x} \in \mathbb{R}^n \text{ is optimal for (LP)} \\ \qquad \text{iff} \\ \exists\, \hat{y} \in \mathbb{R}^m \text{ s.t.} \quad (1) \quad c^T\hat{x} = b^T\hat{y} \\ \qquad\qquad\qquad (2) \quad A^T\hat{y} = c \\ \qquad\qquad\qquad (3) \quad \hat{y} \geqslant 0 \\ \qquad\qquad\qquad (4) \quad A\hat{x} \geqslant b \end{cases}$$

Now we show that (1), (2), (3), (4) are just the KKT conditions for (P).

Indeed substituting A, b, c we obtain

(1) $\nabla f(\hat{x})\,\hat{x} = -[\nabla g_1(\hat{x})^T \dots \nabla g_m(\hat{x})^T]\,\hat{y} + g(\hat{x})^T\hat{y}$.

(2) $-[\nabla g_1(\hat{x})^T \dots \nabla g_m(\hat{x})^T]\,\hat{y} = \nabla f(\hat{x})^T$

(3) $\hat{y} \geq 0$

(4) $-\begin{bmatrix} \nabla g_1(\hat{x}) \\ \vdots \\ \nabla g_m(\hat{x}) \end{bmatrix}\hat{x} \geq -\begin{bmatrix} \nabla g_1(\hat{x}) \\ \vdots \\ \nabla g_m(\hat{x}) \end{bmatrix}\hat{x} + g(\hat{x})$ i.e., $g(\hat{x}) \leq 0$

Consider the KKT-conditions for $(\hat{x}, \hat{y}) \in \mathbb{R}^n \times \mathbb{R}^m$:

(1') $\nabla f(\hat{x}) + y^T \begin{bmatrix} \nabla g_1(\hat{x}) \\ \vdots \\ \nabla g_m(\hat{x}) \end{bmatrix} = 0$

(2') $g(\hat{x}) \leq 0$

(3') $y \geq 0$

(4') $y^T g(\hat{x}) = 0$.

(**) <u>Claim</u>: Let $(\hat{x}, \hat{y}) \in \mathbb{R}^n \times \mathbb{R}^m$. Then (1), (2), (3), (4) hold iff (1'), (2'), (3'), (4') hold.

<u>Proof</u>. Suppose (1), (2), (3), (4) hold.
Then (3') = (3) holds.
Also (2') = (4) holds.
Moreover (1') = (2) holds.
But (1), (2) imply $g(\hat{x})^T\hat{y} = 0$ i.e., $y^T g(\hat{x}) = 0$.
So (4') holds.

Now suppose (1'), (2'), (3'), (4') hold.
Then (2) = (1') holds;
(3) = (3') holds,
(4) = (2') holds.
Also (4'), (1') give (1). □

The given problem is convex and regular. So

(***) $\hat{x} \in \mathbb{R}^n$ is an optimal solution for (P) iff $\exists\, \hat{y} \in \mathbb{R}^m$ s.t. (1'), (2'), (3'), (4') are satisfied

From (*), (**), (***) it follows that:
$\hat{x} \in \mathbb{R}^n$ is an optimal solution for (P) iff $\hat{x} \in \mathbb{R}^n$ is an optimal solution for (LP).

Exercise 21.17.

Let $f(x) := \frac{1}{2}\|Ax-b\|^2 = \frac{1}{2}(Ax-b)^T(Ax-b)$
$= \frac{1}{2}x^TA^TAx - b^TAx + \frac{1}{2}b^Tb$,

and

$g(x) := b - Ax$.

Then $\nabla f(x) = x^TA^TA - b^TA$,

$$\nabla g(x) = \begin{bmatrix} \nabla g_1(x) \\ \vdots \\ \nabla g_m(x) \end{bmatrix} = -A.$$

The problem is regular (for example, with $x = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$, $g(x) = b - Ax < 0$),

and it is also convex.

So the KKT-conditions are necessary and sufficient for a global minimizer.

(KKT-1) $6x_1 - 3x_2 - 2y_1 + y_2 - y_3 - 7 = 0$
$-3x_1 + 6x_2 + y_1 - 2y_2 - y_3 - 4 = 0$,

(KKT-2) $2 - 2x_1 + x_2 \leq 0$
$1 + x_1 - 2x_2 \leq 0$
$4 - x_1 - x_2 \leq 0$

(KKT-3) $y_1, y_2, y_3 \geqslant 0$.

(KKT-4) $y_1(2 - 2x_1 + x_2) = 0$
$y_2(1 + x_1 - 2x_2) = 0$
$y_3(4 - x_1 - x_2) = 0$.

With $x_1 = \frac{13}{6}$ and $x_2 = \frac{11}{6}$, we have

(KKT-2): $\overbrace{2 - \frac{26}{6} + \frac{11}{6}}^{2-2x_1+x_2} = 2 - \frac{15}{6} = \frac{12-15}{6} = -\frac{3}{6} = -\frac{1}{2} < 0$

So $y_1 = 0$ (from KKT-3)

$1 + x_1 - 2x_2 = 1 + \frac{13}{6} - \frac{22}{6} = 1 - \frac{9}{6} = \frac{-3}{6} = -\frac{1}{2} < 0$.

So $y_2 = 0$ (from KKT-3).

$4 - x_1 - x_2 = 4 - \frac{13}{6} - \frac{11}{6} = 4 - \frac{24}{6} = 4 - 4 = 0$,

So (KKT-2) holds.

(KKT-1) gives: $6x_1 - 3x_2 - 2y_1 + y_2 - y_3 - 7 = 0$

i.e., $6 \cdot \frac{13}{6} - 3 \cdot \frac{11}{6} - y_3 - 7 = 0$

and so $y_3 = \frac{1}{2}$.

So with $y_1 = y_2 = 0$ and $y_3 = \frac{1}{2}$, the first equation in (KKT-1) is satisfied.

Also then

$$-3x_1 + 6x_2 + y_1 - 2y_2 - y_3 - 4$$
$$= -3 \cdot \frac{13}{6} + 6 \cdot \frac{11}{6} - \frac{1}{2} - 4 = \frac{9}{2} - 4 - \frac{1}{2} = 0.$$

So (KKT-1) holds.

(KKT-3) holds since $y_1 = y_2 = 0 \geqslant 0$ and $y_3 = \frac{1}{2} \geqslant 0$.

(KKT-4) holds since $y_1 = y_2 = 0$ and

$y_3(4 - x_1 - x_2) = \frac{1}{2} \cdot 0 = 0$.

Thus all the KKT-optimality conditions hold with

$x = \left(\frac{13}{6}, \frac{11}{6}\right)$ and $y = \left(0, 0, \frac{1}{2}\right)$.

So $x = \left(\frac{13}{6}, \frac{11}{6}\right)$ is a global optimal solution for this problem.

Exercise 21.18

(1) Let $f(x) := c_1 x_1 - 4x_2 - 2x_3$,

$$g_1(x) := x_1^2 + x_2^2 - 2$$
$$g_2(x) := x_1^2 + x_3^2 - 2$$
$$g_3(x) := x_2^2 + x_3^2 - 2.$$

Then the problem becomes: $\begin{cases} \text{minimize} & f(x), \\ \text{subject to} & g_i(x) \leq 0, \ i = 1,2,3 \end{cases}$

Since f and the g_is are convex, the given problem is a convex optimization problem. (since f is linear and the Hessians of g_1, g_2, g_3 are positive semidefinite $\forall x$)

(2) (KKT-1) $\exists\, y_1, y_2, y_3$ s.t. $\nabla f(x) + y_1 \nabla g_1(x) + y_2 \nabla g_2(x) + y_3 \nabla g_3(x) = 0$

i.e.,
$$c_1 + 2y_1 x_1 + 2y_2 x_1 = 0$$
$$-4 + 2y_1 x_2 + 2y_3 x_2 = 0$$
$$-2 + 2y_2 x_3 + 2y_3 x_3 = 0$$

✓ (KKT-2) $g_i(x) \leq 0$ for $i = 1,2,3$,

i.e.,
$$x_1^2 + x_2^2 \leq 2$$
$$x_1^2 + x_3^2 \leq 2$$
$$x_2^2 + x_3^2 \leq 2.$$

✓ (KKT-3) $y_1 \geq 0, \ y_2 \geq 0, \ y_3 \geq 0$ ✓

✓ (KKT-4) $y_i g_i(x) = 0, \quad i = 1,2,3$

i.e.,
$$y_1 (x_1^2 + x_2^2 - 2) = 0 \quad ✓$$
$$y_2 (x_1^2 + x_3^2 - 2) = 0 \quad ✓$$
$$y_3 (x_2^2 + x_3^2 - 2) = 0. \quad ✓$$

(3) With $x = \left(\frac{7}{5}, \frac{1}{5}, \frac{1}{5}\right)$, we have
$$x_1^2 + x_2^2 = \frac{49}{25} + \frac{1}{25} = 2 \leq 2$$
$$x_1^2 + x_3^2 = 2 \leq 2$$
$$x_2^2 + x_3^2 = \frac{2}{25} < 2$$

So (KKT-2) is satisfied. Also $x_2^2 + x_3^2 - 2 \neq 0$ and so $y_3 = 0$ from (KKT-4). From (KKT-1),
$$-2 + 2 \cdot y_2 \cdot \frac{1}{5} = 0 \quad \text{and so} \quad y_2 = 5 \geq 0$$
$$-4 + 2 \cdot y_1 \cdot \frac{1}{5} = 0 \quad \text{and so} \quad y_1 = 10 \geq 0$$

So (KKT-3) holds and (KKT-4) holds as well. Finally,
$$c_1 + 2 \cdot \cancel{10}^{2} \cdot \frac{7}{\cancel{5}_1} + 2 \cdot \cancel{5} \cdot \frac{7}{\cancel{5}} = 0 \quad \text{i.e.,} \quad c_1 = -42.$$

So (KKT-1) is satisfied iff $c_1 = -42$.

The problem is regular since for example with $x = 0$, we have $g_1(x) = g_2(x) = g_3(x) = -2 < 0$.
So the KKT-conditions are necessary and sufficient for a global optimal solution. Since with $x = (\frac{7}{5}, \frac{1}{5}, \frac{1}{5})$ the KKT-conditions are satisfied iff $c_1 = -42$, it follows that $(\frac{7}{5}, \frac{1}{5}, \frac{1}{5})$ is an optimal solution to the problem iff $c_1 = -42$.

(4) With $x = (1,1,1)$ we have $x_1^2 + x_2^2 = x_2^2 + x_3^2 = x_3^2 + x_1^2 = 2$. So (KKT-4) holds.
So (KKT-2) is satisfied. (KKT-1) gives:

$$\left.\begin{aligned} c_1 + 2y_1 + 2y_2 &= 0 \\ -4 + 2y_1 + 2y_3 &= 0 \\ -2 + 2y_2 + 2y_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 + 2y_1 + 2y_2 &= 0 \\ -2 + 2y_1 - 2y_2 &= 0 \\ c_1 - 2 + 4y_1 &= 0 \qquad y_1 = \frac{2-c_1}{4} \end{aligned}$$

$$y_2 = \frac{1}{2}\left(-2 + \cancel{2}\frac{(2-c_1)}{\cancel{4}2}\right) = \frac{1}{2}\left(\frac{-4+2-c_1}{2}\right) = \frac{-2-c_1}{4}$$

Finally $y_3 = \frac{1}{2}\left(2 + 2 \cdot \left(\frac{2+c_1}{4}\right)\right) = 1 + \frac{2+c_1}{4} = \frac{6+c_1}{4}$

So (KKT-1) is satisfied iff $(y_1, y_2, y_3) = \left(\frac{2-c_1}{4}, \frac{-2-c_1}{4}, \frac{6+c_1}{4}\right)$.
(KKT-3) is now satisfied iff

$2 - c_1 \geq 0$ i.e., $c_1 \leq 2$
$-2 - c_1 \geq 0$ i.e., $c_1 \leq -2$
$6 + c_1 \geq 0$ i.e., $c_1 \geq -6$

i.e., iff $-6 \leq c_1 \leq -2$.

Thus $(1,1,1)$ is optimal iff $-6 \leq c_1 \leq -2$.

Exercise 21.19

(1) f is convex, since its Hessian is positive definite:

$$F(x) = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(for example using Sylvester's criterion).

So $\hat{x}$ is a global minimizer iff $\nabla f(\hat{x}) = 0$.

$\nabla f(\hat{x}) = 0$ is equivalent to
$$\begin{cases} 2\hat{x}_1 - \hat{x}_2 = 2 \\ -\hat{x}_1 + 2\hat{x}_2 = -4 \\ 2\hat{x}_3 = 0 \end{cases}$$

and so $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3) = (0, -2, 0)$

(2) (KKT-1) : $\nabla f(x) + y_1 \nabla g_1(x) + y_2 \nabla g_2(x) = 0$.

i.e,
$$2x_1 - x_2 - 2 + y_1(-1) + y_2(0) = 0.$$
$$-x_1 + 2x_2 + 4 + y_1(-1) + y_2(0) = 0$$
$$2x_3 + y_1(0) + y_2(-1) = 0.$$

When $x = (1, -1, 1)$ we have:

$+1 - y_1 = 0$ and so $y_1 = 1$

$y_2 = 2$.

So (KKT-1) is satisfied with $y_1 = 1$ and $y_2 = 2$.

(KKT-2). $g_1(1, -1, 1) = -1 + 1 = 0 \leq 0$.

$g_2(1, -1, 1) = 1 - 1 = 0 \leq 0$.

So (KKT-2) is satisfied.

(KKT-3) $y_1 = 1 \geq 0$.

$y_2 = 2 \geq 0$.

So (KKT-3) is satisfied.

(KKT-4) $y_1 g_1(x) = 1 \cdot 0 = 0$.

$y_2 g_2(x) = 2 \cdot 0 = 0$.

So (KKT-4) is satisfied.

Thus the KKT-conditions are satisfied with $x = (1, -1, 1)$ and $y = (1, 2)$.

Exercise 21.20

(1) $x \xrightarrow{\varphi} -(x_1+x_2)$ is convex.

$y \xrightarrow{\psi} e^y$ is increasing.

Thus $x \xrightarrow{\psi\circ\varphi} e^{-(x_1+x_2)}$ is convex.

Also, since $x \mapsto e^{x_1}$ and $x \mapsto e^{x_2}$ are convex, so is

$x \mapsto e^{x_1} + e^{x_2} - 20$.

Finally, $x \mapsto -x_1$ is convex.

Thus f, g_1, g_2 defined by

$$f(x) = e^{-(x_1+x_2)}$$

$$g_1(x) = e^{x_1} + e^{x_2} - 20$$

$$g_2(x) = -x_1,$$

are all convex.

Hence the given problem

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & g_1(x) \le 0, \\ & g_2(x) \le 0 \end{cases}$$

is a convex optimization problem.

Also, $g_1(1,0) = e^1 + 1 - 20 < 0$ and $g_2(1,0) = -1 < 0$.

So the problem is regular as well.

(2) For a regular convex problem, x is optimal iff the KKT-conditions hold, that is, that there exists a $y \in \mathbb{R}^2$ such that the following hold:

(KKT-1) $\nabla f(x) + y^T \nabla g(x) = 0$ that is

$$\begin{bmatrix} e^{-(x_1+x_2)} & -e^{-(x_1+x_2)} \end{bmatrix} + \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} e^{x_1} & e^{x_2} \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

So $e^{-(x_1+x_2)} - y_1 e^{x_1} + y_2 = 0$ and $e^{-(x_1+x_2)} - y_1 e^{x_2} = 0$.

(KKT-2) $g_i(x) \le 0$ for all i, that is, $x_1 \geqslant 0$ and $e^{x_1} + e^{x_2} \le 20$.

(KKT-3) $\quad y \geq 0$, that is, $y_1 \geq 0$ and $y_2 \geq 0$.

(KKT-4) $\quad y_i g_i(x) = 0$ for all i, that is,

$y_1 (e^{x_1} + e^{x_2} - 20) = 0$ and $y_2 x_1 = 0$.

If $x_1 = 0$, then (KKT-1) gives $y_1 = e^{-2x_2} \neq 0$. (KKT-4) then gives that $e^{x_1} + e^{x_2} - 20 = 0$, and since $x_1 = 0$, we further obtain that $e^{x_2} = 19$. (KKT-1) gives $y_2 = e^{-2x_2} - e^{-x_2} = \frac{1}{19}\left(\frac{1}{19} - 1\right) < 0$, contradicting (KKT-3). So it cannot be the case that $x_1 = 0$.

If $x_1 \neq 0$, then (KKT-4) gives $y_2 = 0$. (KKT-1) then gives first of all that $y_1 = e^{-x_1 - 2x_2} > 0$. Also, $y_1 (e^{x_1} - e^{x_2}) = 0$ and since $y_1 > 0$, we obtain $e^{x_1} = e^{x_2}$, which implies that $x_1 = x_2$. (KKT-4) together with $y_1 > 0$ gives $e^{x_1} + e^{x_2} - 20 = 0$. Since $x_1 = x_2$ we now obtain that $e^{x_1} = e^{x_2} = 10$, so that $x_1 = x_2 = \log_e 10$. Then it is easily verified that (KKT-1) to (KKT-4) hold with $x_1 = x_2 = \log_e 10$, $y_1 = e^{-x_1 - 2x_2} = e^{-3\log_e 10} = \frac{1}{1000}$ and $y_2 = 0$. So the global optimal solution is given by

$$x_1 = x_2 = \log_e 10.$$