

Exercise 25.4

$$\begin{aligned}
 (1) \quad \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 2 \end{bmatrix} &\xrightarrow{\frac{1}{2}r_1} \begin{bmatrix} 1 & 1/2 & -1/2 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow{r_2 - 3r_1} \begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & -1/2 & 7/2 \end{bmatrix} \\
 &\xrightarrow{-2r_2} \begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & 1 & -7 \end{bmatrix} \\
 &\xrightarrow{r_1 - 1/2 r_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -7 \end{bmatrix}
 \end{aligned}$$

Thus we define $C := U := \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -7 \end{bmatrix}$, and

$$B := A_\beta := \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}.$$

(Then $BC = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 2 \end{bmatrix} = A.$)

$$\begin{aligned}
 (2) \quad \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{bmatrix} &\xrightarrow{\substack{r_2 - 2r_1 \\ r_3 - 3r_1}} \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & -4 \end{bmatrix} \xrightarrow{-\frac{1}{2}r_2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -4 \end{bmatrix} \xrightarrow{\substack{r_1 - r_2 \\ r_3 + 4r_2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.
 \end{aligned}$$

Thus we define $C := U := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and

$$B := A_\beta := \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{bmatrix}.$$

(Then $BC = A.$)

Exercise 25.5

(See the solution to Exercise 25.4.)

(1) A basis for

$$\text{ran} \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 2 \end{bmatrix} \text{ is given by } \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

A basis for

$$\text{ran} \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ -1 & 2 \end{bmatrix} \text{ is given by } \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -7 \end{bmatrix} \right\}.$$

(2) A basis for

$$\text{ran} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{bmatrix} \text{ is given by } \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

A basis for

$$\text{ran} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -1 \end{bmatrix} \text{ is given by } \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

Exercise 25.6

(See the solution to Exercise 25.4.)

$$(1) A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 2 \end{bmatrix}. \quad \text{Here } k = n - r = 3 - 2 = 1.$$

$$\ker A = \left\{ x \in \mathbb{R}^3 : x_{\beta} = -U_{\beta} x_{\beta} \right\}.$$

$$x_{\beta} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = - \begin{bmatrix} 3 \\ -7 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}.$$

$$\text{Hence } \ker A = \text{span} \begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix}.$$

$$(2) A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{bmatrix}}_{A_{\beta}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_U.$$

$$\text{Here } k = n - r = 2 - 2 = 0.$$

$$\text{So } \ker A = \{0\}.$$

Exercise 25.7

$$(1) \quad A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 2 \end{bmatrix};$$

$$A^T = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ -1 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}r_1} \begin{bmatrix} 1 & 3/2 \\ 1 & 1 \\ -1 & 2 \end{bmatrix} \xrightarrow{\substack{r_2 - r_1 \\ r_3 + r_1}} \begin{bmatrix} 1 & 3/2 \\ 0 & -1/2 \\ 0 & 7/2 \end{bmatrix} \xrightarrow{-2r_2} \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \\ 0 & 7/2 \end{bmatrix}$$

$$\xrightarrow{\substack{r_1 - \frac{3}{2}r_2 \\ r_3 - \frac{7}{2}r_2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\text{So } A^T = \underbrace{\begin{bmatrix} 2 & 3 \\ 1 & 1 \\ -1 & 2 \end{bmatrix}}_{A_\beta} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_U$$

$$\ker A^T = \ker U = 0.$$

$$(2) \quad A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{bmatrix}; \quad A^T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix}.$$

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \end{bmatrix} \xrightarrow{-\frac{1}{2}r_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_1 - 2r_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\text{So } A^T = \underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}}_{A_\beta} \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}}_U.$$

$$\text{We have } \dim(\ker A^T) = n - r = 3 - 2 = 1.$$

$$\ker A^T = \{x \in \mathbb{R}^3 : x_\beta = -U_\beta x_\nu\}$$

$$x_\beta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = - \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot 1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

$$\text{Hence } \ker A^T = \text{span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

Exercise 25.9

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix} \xrightarrow{\substack{r_2 - r_1 \\ r_3 - 2r_1}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}r_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{\substack{r_3 - r_2 \\ r_3 - 2r_2}} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $A = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 2 & 4 \end{bmatrix}}_{A_\beta} \underbrace{\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}}_U$.

So a basis for $\text{ran } A$ is $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 2 & 4 \end{bmatrix} \right\}$, and

a basis for $\text{ran } A^T$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{1}{2} \end{bmatrix} \right\}$.

We have $\dim(\ker A) = n - r = 3 - 2 = 1$.

$$\ker A = \{x \in \mathbb{R}^3 : x_\beta = -U_\beta x_U\}$$

$$x_\beta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \cdot 1 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}. \text{ Hence } \ker A = \text{span} \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}.$$

$$A^T = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{\substack{r_2 - r_1 \\ r_3 - r_1}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}r_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\substack{r_1 - r_2 \\ r_3 - r_2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $A^T = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Again $\dim(\ker A^T) = n - r = 3 - 2 = 1$

$$\ker A^T = \{x \in \mathbb{R}^3 : x_\beta = -U_\beta x_U\}$$

$$x_\beta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}. \text{ Hence } \ker A^T = \text{span} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

