KTH Mathematics

SF2822 Applied nonlinear optimization, final exam
Friday August 17 2012 8.00–13.00
Brief solutions

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1. (a) As $g(x^*) = 0$, the constraint is active, and as $\nabla g(x^*)$ is nonzero, it holds that x^* is a regular point. Hence, for x^* to be a local minimizer to (NLP), the first-order necessary optimality conditions must hold. Hence, there must exist a nonnegative λ^* such that $\nabla f(x^*) = \nabla g(x^*)\lambda^*$, i.e.,

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \lambda^*.$$

There is no such λ^*. Hence, x^* is not a local minimizer to (NLP).

(b) The first-order optimality conditions

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \lambda^*,$$

are only violated in the last component if $\lambda^* = 1$. Hence, for this value of λ^*, we have

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \lambda^* + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Consequently, if we add a second constraint in the form of the bound-constraint $-x_3 \geq -x_3^*$ to (NLP), the first-order optimality conditions are satisfied for $\lambda_1^* = 1$, $\lambda_2^* = 1$.

In order to verify if x^* is a local minimizer, we now examine the second-order optimality conditions. We obtain

$$\begin{aligned} \nabla^2 \mathcal{L}(x^*, \lambda^*) &= \nabla^2 f(x^*) - \nabla^2 g(x^*)\lambda_1^* \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}. \end{aligned}$$

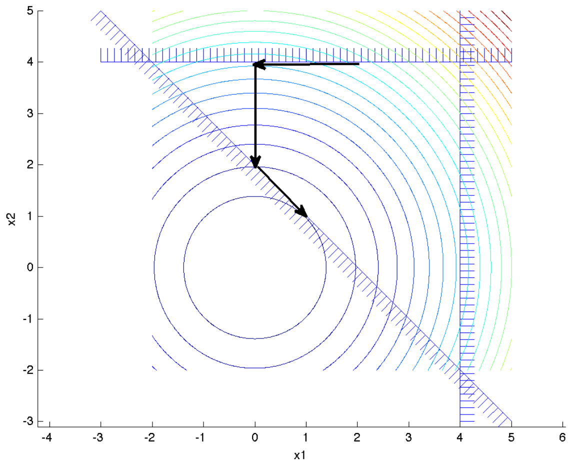
As we have strict complementarity, we now want to check the definiteness of the reduced Hessian of the Lagrangian with respect to the active constraint gradients, given by

$$A(x^*) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

However, $\nabla^2\mathcal{L}(x^*, \lambda^*)$ is a diagonal matrix with positive diagonal elements, hence positive definite. Thus, the reduced Hessian is also positive definite.

We conclude that the second-order sufficient optimality conditions hold at x^* together with λ^*. Hence, x^* is a local minimizer to the problem where the bound-constraint $-x_3 \geq -x_3^*$ has been added.

2. (a) The iterations are illustrated in the figure below.

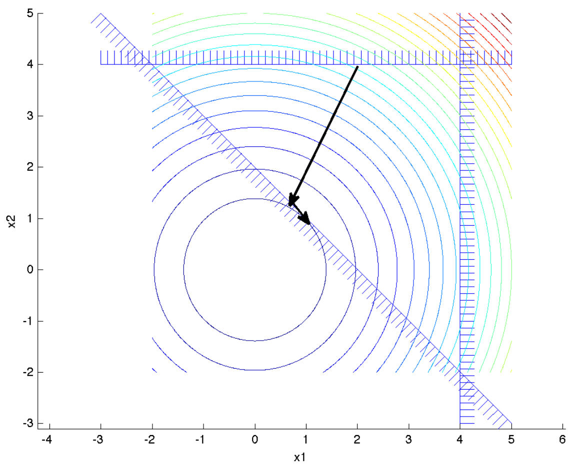


In the first iteration, the search direction points towards the minimizer with constraint 2 active, $(0\ 4)^T$, and the step is not limited. Hence, the next iterate is $(0\ 4)^T$ at which the multiplier of constraint 2 is negative. -4. Thus, constraint 2 is deleted.

In the second iteration, the search direction points towards the unconstrained minimizer, the origin. The step is limited by constraint 3 at $(0\ 2)^T$. Hence, the next iterate is $(0\ 2)^T$ and constraint 3 is added.

In the third iteration, the search direction points towards the minimizer with constraint 3 active, $(1\ 1)^T$, and the step is not limited. Hence, the next iterate is $(1\ 1)^T$ at which the multiplier of constraint 3 is positive. 1. Thus, the optimal solution has been found.

(b) The iterations are illustrated in the figure below.



In the first iteration, the search direction points towards the unconstrained minimizer, the origin. The step is limited by constraint 3 at $(2/3\ 4/3)^T$. Hence, the next iterate is $(2/3\ 4/3)^T$ with constraint 3 active.

In the second iteration, the search direction points towards the minimizer with constraint 3 active, $(1\ 1)^T$, and the step is not limited. Hence, the next iterate is $(1\ 1)^T$ at which the multiplier of constraint 3 is positive. 1. Thus, the optimal solution has been found.

3. (a) We have

$$f(x) = 4(x_1-2)^2 + (x_2-1)^2 \qquad g(x) = 1 - x_1^2 - x_2^2 \geq 0,$$
$$\nabla f(x) = \begin{pmatrix} 8(x_1-2) \\ 2(x_2-1) \end{pmatrix}, \qquad \nabla g(x) = \begin{pmatrix} -2x_1 \\ -2x_2 \end{pmatrix},$$
$$\nabla^2 f(x) = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \nabla^2 g(x) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Since $g(x^{(0)}) > 0$, we need not introduce slack variables. This could be done, but we choose not to do so. The primal-dual nonlinear equation to be solved is then

$$\begin{aligned} \nabla f(x) - \nabla g(x)\lambda &= 0, \\ g(x)\lambda - \mu &= 0. \end{aligned}$$

The Newton step $\Delta x^{(0)}$, $\Delta\lambda^{(0)}$ is given by

$$\begin{pmatrix} \nabla^2_{xx}\mathcal{L}(x^{(0)},\lambda^{(0)}) & \nabla g(x^{(0)}) \\ \lambda^{(0)}\nabla g(x^{(0)})^T & -g(x^{(0)}) \end{pmatrix} \begin{pmatrix} \Delta x^{(0)} \\ -\Delta\lambda^{(0)} \end{pmatrix} = - \begin{pmatrix} \nabla f(x^{(0)}) - \nabla g(x^{(0)})\lambda^{(0)} \\ g(x^{(0)})\lambda^{(0)} - \mu^{(0)} \end{pmatrix}.$$

Insertion of numerical values gives

$$f(x^{(0)} = 17, \qquad g(x^{(0)}) = 1,$$
$$\nabla f(x^{(0)}) = \begin{pmatrix} -16 \\ -2 \end{pmatrix}, \qquad \nabla g(x^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$\nabla^2 f(x^{(0)} = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \nabla^2 g(x^{(0)} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix},$$
$$\nabla^2 \mathcal{L}(x^{(0)},\lambda^{(0)}) = \begin{pmatrix} 10 & 0 \\ 0 & 4 \end{pmatrix},$$

so that the linear equations become

$$\begin{pmatrix} 10 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \Delta x_1^{(0)} \\ \Delta x_2^{(0)} \\ -\Delta\lambda^{(0)} \end{pmatrix} = \begin{pmatrix} 16 \\ 2 \\ 0 \end{pmatrix}.$$

(b) We may ensure that the iterates $x^{(1)}$ and $\lambda^{(1)}$ remain interior by selecting a steplength $\alpha^{(0)}$ such that $x^{(1)} = x^{(0)} + \alpha^{(0)}\Delta x^{(0)}$ and $\lambda^{(1)} = \lambda^{(0)} + \alpha^{(0)}\Delta\lambda^{(0)}$ such that $g(x^{(1)}) > 0$ and $\lambda^{(1)} > 0$.

4. (See the course material.)

5. (a) Let $x^{(k)}$, $\lambda^{(k)}$, be a given iterate. Since (NLP) has at least one feasible point, there is an $\bar{x}$ such that $g_i(\bar{x}) \geq 0$, $i = 1, \dots, m$.
If g_i is concave on $I\!R^n$, it holds that

$$0 \leq g_i(\bar{x}) \leq g_i(x^{(k)}) + \nabla g_i(x^{(k)})^T(\bar{x} - x^{(k)}).$$

Hence, if all the constraint functions are concave, it follows that

$$g_i(x^{(k)}) + \nabla g_i(x^{(k)})^T(\bar{x} - x^{(k)}) \geq 0, \quad i = 1, \dots, m.$$

As the constraints of the quadratic programming subproblem are given by

$$g_i(x^{(k)}) + \nabla g_i(x^{(k)})^T \Delta x \geq 0, \quad i = 1, \dots, m,$$

it follows that $\Delta x = \bar{x} - x^{(k)}$ is feasible to the quadratic programming subproblem.

(b) The requirement $g_1(x) \geq 0$ gives $x_1 \geq 1$ or $x_1 \leq -1$. Hence, taking g_2 and g_3 into account, it follows that the feasible region of (NLP) is given by $1 \leq x \leq 2$. Thus, (NLP) has feasible points.
Differentiation gives the cconstraints for the first quadratic programming subproblem as

$$\begin{pmatrix} 2x^{(0)} \\ 1 \\ -1 \end{pmatrix} \Delta x \geq - \begin{pmatrix} x^{(0)2} - 1 \\ x^{(0)} \\ -x^{(0)} + 2 \end{pmatrix}.$$

Insertion of numerical values gives

$$\begin{aligned} 0 &\geq 1, \\ \Delta x &\geq 0, \\ \Delta x &\leq 2, \end{aligned}$$

where the first equation is not compatible, so that there is no feasible solution to the first quadratic programming subproblem.
(For this small problem it is easy to detect that the constraints can be formulated in a much simpler way, $1 \leq x \leq 2$. In general, we cannot expect to do so for a large-scale problem. This particular example is very special, but an SQP solver for nonconvex problems must be able to handle infeasible subproblems somehow. There are various techniques for doing so.)