KTH Mathematics

SF2822 Applied nonlinear optimization, final exam
Monday May 20 2013 8.00–13.00
Brief solutions

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1. (a) If p is a nonzero vector in $I\!R^4$, then

$$p^T H p = p^T (I + ee^T) p = p^T p + (p^T e)^2 \ge p^T p > 0.$$

Hence, H is positive definite.

(b) We may write $A = (N\ B)$, where $B = I$ and $N = -e$, where e is the vector of ones. Then, a matrix Z whose columns form a basis for the nullspace of A is given by

$$Z = \begin{pmatrix} I \\ -B^{-1}N \end{pmatrix} = \begin{pmatrix} 1 \\ e \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

(c) The step to the minimizer is given by Zp_Z, where

$$Z^T H Z p_Z = -Z^T (H\bar{x} + c).$$

Insertion of numerical values gives $20 p_Z = 20$, i.e., $p_Z = 1$. Hence the optimal x is given by

$$x = \bar{x} + Z p_Z = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

The Lagrange multipliers are then given by $Hx + c = A^T \lambda$, i.e.,

$$\begin{pmatrix} -2 \\ -1 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix},$$

i.e., $\lambda = (-1 \quad -1 \quad 4)^T$.

(d) Since H is positive definite, the optimal solution x is unique. As A has full row rank, the Lagrange multiplier vector λ is unique. Since no component of λ is zero, no constraint can be omitted without x being changed.

2. Constraint 3 is in the working set at the initial point, i.e., $\mathcal{W} = \{3\}$. With $H = I$ and $c = 0$ we obtain

$$\begin{pmatrix} H & A_{\mathcal{W}}^T \\ A_{\mathcal{W}} & 0 \end{pmatrix} \begin{pmatrix} p^{(0)} \\ -\lambda_{\mathcal{W}}^{(0)} \end{pmatrix} = - \begin{pmatrix} Hx^{(0)} + c \\ 0 \end{pmatrix}.$$

Insertion of numeric values gives

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} p_1^{(0)} \\ p_2^{(0)} \\ -\lambda_3^{(1)} \end{pmatrix} = - \begin{pmatrix} 8 \\ 0 \\ 0 \end{pmatrix}$$

We obtain

$$p^{(0)} = \begin{pmatrix} -8 & 0 \end{pmatrix}^T, \quad \lambda^{(1)} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T.$$

The maximum steplength is given by

$$\alpha_{\max} = \min_{i:a_i^T p^{(0)}<0} \frac{a_i^T x^{(0)} - b_i}{-a_i^T p^{(0)}} = \frac{1}{4},$$

where the minimium is attained for $i = 1$. Consequently, $\alpha^{(0)} = 1/4$ so that

$$x^{(1)} = x^{(0)} + \alpha^{(0)} p^{(0)} = \begin{pmatrix} 8 \\ 0 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} -8 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix},$$

with $\mathcal{W} = \{1, 3\}$. The solution to the corresponding equality-constrained quadratic progam is given by

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_1^{(1)} \\ p_2^{(1)} \\ -\lambda_1^{(2)} \\ -\lambda_3^{(2)} \end{pmatrix} = - \begin{pmatrix} 6 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We obtain

$$p^{(1)} = \begin{pmatrix} 0 & 0 \end{pmatrix}^T, \quad \lambda^{(2)} = \begin{pmatrix} 6 & 0 & -6 \end{pmatrix}^T.$$

As $p^{(1)} = 0$, it follows that $x^{(2)} = x^{(1)}$ and the corresponding equality-constrained problem has been solved. However, since $\lambda_3^{(2)} < 0$, constraint 3 is deleted so that $\mathcal{W} = \{1\}$. The solution to the corresponding equality-constrained quadratic progam is given by

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} p_1^{(2)} \\ p_2^{(2)} \\ -\lambda_1^{(3)} \end{pmatrix} = - \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}.$$

We obtain

$$p^{(2)} = \begin{pmatrix} -3 & 3 \end{pmatrix}^T, \quad \lambda^{(3)} = \begin{pmatrix} 3 & 0 & 0 \end{pmatrix}^T.$$

The maximum steplength is given by

$$\alpha_{\max} = \min_{i:a_i^T p^{(2)}<0} \frac{a_i^T x^{(2)} - b_i}{-a_i^T p^{(2)}} = \frac{4}{3},$$

where the minimium is attained for $i = 2$. Since $\alpha_{\max} > 1$, we let $\alpha^{(2)} = 1$ so that

$$x^{(3)} = x^{(2)} + p^{(2)} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Since $\lambda^{(3)} \geq 0$, the optimal solution has been found.

3. (a) The problem (QP) is a convex quadratic program. The primal part of the trajectory is obtained as minimizer to the barrier-transformed problem

$$(P_\mu) \qquad \min \quad \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2 - \mu \ln(x_1 - 1)$$

under the implicit condition that $x_1 + 1 > 0$. The first-order optimality conditions of (P_μ) gives

$$x_1(\mu) - \frac{\mu}{x_1(\mu) - 1} = 0,$$
$$x_2(\mu) = 0.$$

Since (QP) is a convex problem, (P_μ) is an unconstrained convex problem, taking into account the implicit constraint $x_1 - 1 > 0$. Therefore, the first-order necessary optimality conditions are sufficient for global optimality.

The first-order optimality conditions give $x_2(\mu) = 0$, and $x_1(\mu)$ is given by

$$x_1^2(\mu) - x_1(\mu) - \mu = 0,$$

i.e.,

$$x_1(\mu) = \frac{1}{2} + \sqrt{\frac{1}{4} + \mu},$$

where the plus sign has been chosen for the square root to enforce $x_1(\mu) - 1 > 0$. The dual part of the trajectory, i.e. $\lambda(\mu)$, is normally given by $\lambda_i(\mu) = \mu/g_i(x(\mu))$, $i = 1, \ldots, m$. Here we only have one constraint, so

$$\lambda(\mu) = \frac{\mu}{x_1(\mu) - 1} = \frac{\mu}{-\frac{1}{2} + \sqrt{\frac{1}{4} + \mu}} = \frac{1}{2} + \sqrt{\frac{1}{4} + \mu}.$$

(b) As $\mu \to 0$ it follows that $x(\mu) \to (1\ 0)^T$ and $\lambda(\mu) \to 1$. Let $x^* = (1\ 0)^T$ and $\lambda^* = 1$. Then x^* and λ^* satisfy the first-order optimality conditions of (QP). Since (QP) is a convex problem, this is sufficient for global optimality of (QP).

(c) We have

$$\|x(\mu) - x^*\|_2 = -\frac{1}{2} + \sqrt{\frac{1}{4} + \mu} = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\mu} = \mu + o(\mu).$$

This is as expected. We would expect $\|x(\mu) - x^*\|_2$ to be of the order μ near an optimal solution where regularity holds.

4. (a) The point $x^{(0)}$ is not feasible, as $g_2(x^{(0)}) < 0$. Hence, it cannot be a local minimizer.

(b) The QP subproblem becomes

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}p^T\nabla^2_{xx}\mathcal{L}(x^{(0)},\lambda^{(0)})p + \nabla f(x^{(0)})^T p \\ \text{subject to} & \nabla g_i(x^{(0)})^T p \geq -g_i(x^{(0)}), \quad i = 1,2,3. \end{array}$$

Insertion of numerical values gives

$$\begin{array}{ll} \min & \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 \\ \text{subject to} & p_1 + p_2 \geq -2, \\ & p_1 \geq 1, \\ & p_2 \geq -1. \end{array}$$

If we let $p^{(0)}$ denote the optimal solution of the QP subproblem, we obtain $x^{(1)} = x^{(0)} + p^{(0)}$. We obtain $\lambda^{(1)}$ as the Lagrange multipliers of the QP subproblem. The quadratic program is convex, and the optimal solution is given by $p^{(0)} = (1\ 0)^T$, so that $x^{(2)} = x^{(0)} + p^{(0)} = (1\ 0)^T$. The Lagrange multiplier of the quadratic program is given by$\lambda^{(1)} = (0\ 1)^T$.

5. (See the course material.)