SF2822 Applied nonlinear optimization, final exam
Wednesday August 15 2018 8.00–13.00
Brief solutions

1. (a) This claim is true.

The output of "`active=find(g<sqrt(eps))`" shows that constraints 1 and 3 are active at x^*.

In addition, the command "`rank(A(active,:))`" gives 2, showing that the constraint gradients of the active constraints are linearly independents, i.e., x^* is a regular point.

(b) This claim is true.

The output of "`norm(gradf-A'*lambdastar)`" is of the order of machine precision, i.e., $\nabla f(x^*) - A(x^*)^T\lambda^*$ is numerically zero.

In addition, the output of "`[g lambdastar]`" shows that $g(x^*) \geq 0$, $\lambda^* \geq 0$ and $g_i(x^*)\lambda_i^* = 0$, $i = 1, \ldots, 24$. Consequently, x^* together with λ^* satisfy the first-order necessary optimality conditions.

(c) This claim is true.

The additional requirement to first-order necessary optimality conditions is that the reduced Hessian $Z^T\nabla^2_{xx}\mathcal{L}(x^*, \lambda^*)Z$ is positive semidefinite. The output of "`eig(Z'*HessL*Z)`" shows that $Z^T\nabla^2_{xx}\mathcal{L}(x^*, \lambda^*)Z$ has all eigenvalues nonnegative, hence being positive semidefinite, where Z is a matrix whose columns form a basis for the nullspace of the Jacobian of the active constraints.

(d) This claim is false.

We have strict complementarity. Hence, in addition to existence of Lagrange multipliers, the second-order sufficient optimality conditions require the reduced Hessian $Z^T\nabla^2_{xx}\mathcal{L}(x^*, \lambda^*)Z$ positive definite. This is not true, since one eigenvalue of $Z^T\nabla^2_{xx}\mathcal{L}(x^*, \lambda^*)Z$ is zero.

(e) This claim is true.

Since constraints $6, 7, \ldots, 24$ are inactive at x^*, they may be omitted from the problem without affecting the local optimality conditions. The resulting problem is then convex, since f and $-g_i$, $i = 1, \ldots, 5$, are convex on $I\!R^9$. Therefore, first-order necessary optimality conditions are sufficient to ensure global minimality.

2. If the problem is put on the form

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \geq 0, \quad x \in I\!R^2, \end{array}$$

we obtain

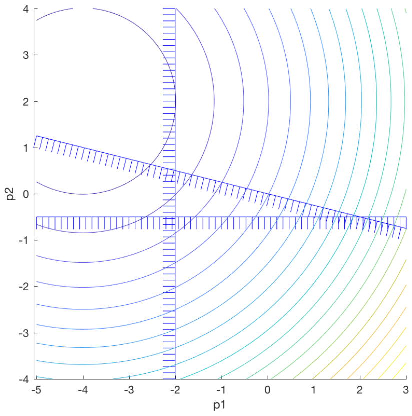
$$\nabla f(x)^T = \begin{pmatrix} x_1 + x_2 + \frac{3}{2} & x_1 + x_2 - \frac{9}{2} \end{pmatrix}, \quad \nabla g(x)^T = \begin{pmatrix} x_2 & x_1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\nabla^2_{xx}\mathcal{L}(x, \lambda) = \begin{pmatrix} 1 & 1 - \lambda_1 \\ 1 - \lambda_1 & 1 \end{pmatrix}.$$

With $x^{(0)} = (2\ \frac{1}{2})^T$ and $\lambda^{(0)} = (1\ 0\ 0)^T$, the first QP-problem becomes

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\begin{pmatrix} p_1 & p_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} 4 & -2 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\ \text{subject to} & \begin{pmatrix} \frac{1}{2} & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ -2 \\ -\frac{1}{2} \end{pmatrix}. \end{array}$$

The optimal solution of the QP-problem is given by the feasible point which is closest, in 2-norm, to $(-4\quad 2)^T$. This may for example be solved graphically:



The solution is $p^{(0)} = (-2\quad 2)^T$ with constraint 2 active. The Lagrange multiplier $\lambda_2^{(1)}$ of the active constraint is given by

$$\begin{pmatrix} -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\lambda_2^{(1)},$$

i.e., $\lambda_2^{(1)} = 2$. Thus, we have $\lambda^{(1)} = (0\ \ 2\ \ 0)^T$, and $x^{(1)}$ is given by $x^{(1)} = x^{(0)} + p^{(0)} = (0\ 5/2)^T$.

3. (a) The problem (QP) is a convex quadratic program. The primal part of the trajectory is obtained as minimizer to the barrier-transformed problem

$$(P_\mu) \qquad \min \quad \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2 - \mu \ln(x_1 - 1)$$

under the implicit condition that $x_1 + 1 > 0$. The first-order optimality conditions of (P_μ) gives

$$x_1(\mu) - \frac{\mu}{x_1(\mu) - 1} = 0,$$
$$x_2(\mu) = 0.$$

Since (QP) is a convex problem, (P_μ) is an unconstrained convex problem, taking into account the implicit constraint $x_1 - 1 > 0$. Therefore, the first-order necessary optimality conditions are sufficient for global optimality.

The first-order optimality conditions give $x_2(\mu) = 0$, and $x_1(\mu)$ is given by

$$x_1^2(\mu) - x_1(\mu) - \mu = 0,$$

i.e.,

$$x_1(\mu) = \frac{1}{2} + \sqrt{\frac{1}{4} + \mu},$$

where the plus sign has been chosen for the square root to enforce $x_1(\mu) - 1 > 0$. The dual part of the trajectory, i.e. $\lambda(\mu)$, is normally given by $\lambda_i(\mu) = \mu / g_i(x(\mu))$, $i = 1, \ldots, m$. Here we only have one constraint, so

$$\lambda(\mu) = \frac{\mu}{x_1(\mu) - 1} = \frac{\mu}{-\frac{1}{2} + \sqrt{\frac{1}{4} + \mu}} = \frac{1}{2} + \sqrt{\frac{1}{4} + \mu}.$$

(b) As $\mu \to 0$ it follows that $x(\mu) \to (1\ 0)^T$ and $\lambda(\mu) \to 1$. Let $x^* = (1\ 0)^T$ and $\lambda^* = 1$. Then x^* and λ^* satisfy the first-order optimality conditions of (QP). Since (QP) is a convex problem, this is sufficient for global optimality of (QP).

(c) We have

$$x_1(\mu) = \lambda(\mu) = \frac{1}{2} + \sqrt{\frac{1}{4} + \mu}, \quad x_1^* = \lambda^* \quad \text{and} \quad x_2(\mu) = x_2^* = 0.$$

Therefore, $\|x(\mu) - x^*\|_2 = \|\lambda(\mu) - \lambda^*\|_2$, and it suffices to consider $\|x(\mu) - x^*\|_2$. The expression from above gives

$$\|x(\mu) - x^*\|_2 = -\frac{1}{2} + \sqrt{\frac{1}{4} + \mu} = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\mu} = \mu + o(\mu).$$

This is as expected. We would expect $\|x(\mu) - x^*\|_2$ and $\|\lambda(\mu) - \lambda^*\|_2$ to be of the order μ near an optimal solution where regularity holds.

4. (a) We may write $A = (I \ \ -e)$, with $e = (1\ 1\ 1\ 1)^T$. Then, a matrix whose columns form a basis for the nullspace of A is given by $Z = (-(-e^T)\ 1)^T = (1\ \ 1\ \ 1\ \ 1\ \ 1)^T$.

(b) The step to the minimizer of the new problem can be written as $p = Zp_Z$, where

$$Z^T H Z p_Z = -Z^T (H x^* + c + 20 e_1).$$

As x^* is optimal to the original problem we have $Z^T(Hx^* + c) = 0$, so that $Z^T H Z p_Z = -20 Z^T e_1$. Insertion of numerical values gives $10 p_z = -20$, i.e., $p_Z = -2$. Hence, if the optimal solution to the new problem is denoted by $\bar{x}$, we obtain

$$\bar{x} = x^* + Z p_z = \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

(c) As $\bar{x}_5 < 0$, $\bar{x}$ is not feasible to the third problem. When finding $\bar{x}$, we computed p as the first step in an active-set method for solving the third problem. The maximum steplength is given by the maximum α such that $x^* + \alpha p \geq 0$. We obtain $\alpha = 1/2$. The new point, $\widehat{x}$, becomes $\widehat{x} = x^* + 1/2 p = (4\ 3\ 2\ 1\ 0)^T$. This point is in fact optimal, as the Lagrange multiplier of an added constraint will become positive. If the constraint $x_5 \geq 0$ is added as a fifth constraint, this can be verified algebraically by solving

$$H\widehat{x} + c = \begin{pmatrix} 20 \\ 1 \\ 2 \\ -5 \\ -8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \\ \hat{\lambda}_3 \\ \hat{\lambda}_4 \\ \hat{\lambda}_5 \end{pmatrix},$$

to obtain the Lagrange multipliers. We obtain $\hat{\lambda}_1 = 20$, $\hat{\lambda}_2 = 1$, $\hat{\lambda}_3 = 2$, $\hat{\lambda}_4 = -5$, $\hat{\lambda}_5 = 10$. As $\hat{\lambda}_5 \geq 0$, the solution is optimal.

5. (See the course material.)