KTH Mathematics

SF2822 Applied nonlinear optimization, final exam
Thursday June 5 2008 8.00–13.00

Examiner: Anders Forsgren, tel. 790 71 27.

Allowed tools: Pen/pencil, ruler and rubber; plus a calculator provided by the department.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

---

1. Consider the nonlinear optimization problem (NLP) defined as

$$(NLP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}(x_1+1)^2 + \frac{1}{2}(x_2+2)^2 \\ \text{subject to} & 3(x_1+x_2-2)^2 + (x_1-x_2)^2 - 6 = 0. \end{array}$$

You have obtained a printout from an SQP solver for this problem. The initial point is $x = (0\ 0)^T$ and $\lambda = 0$. Six iterations, without linesearch, have been performed. The printout reads:

It	x_1	x_2	λ	$\|\nabla f(x) - \nabla g(x)\lambda\|$	$\|g(x)\|$
0	0	0	0	2.2361	6
1	0.75	-0.25	-0.14583	0.74361	1.75
2	0.5285	0.050045	-0.20644	0.098113	0.29052
3	0.57728	0.041731	-0.21804	0.0044016	0.0081734
4	0.57666	0.043089	-0.21854	$4.1731 \cdot 10^{-6}$	$5.5421 \cdot 10^{-6}$
5	0.57666	0.043089	-0.21854	$3.9569 \cdot 10^{-12}$	$4.8512 \cdot 10^{-12}$
6	0.57666	0.043089	-0.21854	$1.1102 \cdot 10^{-15}$	$1.7764 \cdot 10^{-15}$

(a) Formulate the first QP problem. Verify that the solution to this QP problem is given by the printout above. .... (6p)

(b) How would the iterates change if the constraint in (NLP) would be changed to $3(x_1+x_2-2)^2 + (x_1-x_2)^2 - 6 \le 0$? .... (2p)

(c) For the original problem (NLP), show that in this case the iterates converge to a global minimizer. (You need not verify the numerical values.) ...... (2p)

Note: According to the convention of the book we define the Lagrangian $\mathcal{L}(x,\lambda)$ as $\mathcal{L}(x,\lambda) = f(x) - \lambda^T g(x)$, where $f(x)$ the objective function and $g(x)$ is the constraint function.

2. Consider the QP-problem (QP) defined as

$$(QP) \quad \begin{array}{ll} \text{minimize} & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \\ \text{subject to} & x_1 + x_2 \geq 0. \end{array}$$

(a) For a given positive barrier parameter μ, find the corresponding optimal solution $x(\mu)$ and the corresponding multiplier estimate $\lambda(\mu)$ to the barrier-transformed problem. It is possible to obtain an analytical expression for this small problem. .... (5p)

(b) Show that $x(\mu)$ and $\lambda(\mu)$ which you obtained in (2a) converge to the optimal solution and Lagrange multiplier respectively of (QP). .... (3p)

(c) Compute $\|x(\mu) - x^*\|_2$, where x^* denotes the optimal solution to (QP). Is this as expected? Comment on the result. .... (2p)

3. Derive the expression for the symmetric rank-1 update, C_k, in a quasi-Newton update $B_{k+1} = B_k + C_k$. .... (10p)

4. Consider a nonlinear programming problem (NLP) defined by

$$(NLP) \quad \begin{array}{ll} \text{minimize} & e^{x_1} + x_1x_2 + x_2^2 - 2x_2x_3 + x_3^2 \\ \text{subject to} & -x_1^2 - x_2^2 - x_3^2 + 5 \geq 0, \\ & a^Tx + 2 = 0, \end{array}$$

where $a \in I\!R^3$ is a given constant. Let $\widetilde{x} = (0\ 0\ 1)^T$.

(a) Determine a such that $\widetilde{x}$ fulfils the first-order necessary optimality conditions for (NLP). .... (6p)

(b) For the value on a which you determined in (4a), determine if $\widetilde{x}$ is a local minimizer to (NLP). .... (4p)

5. Consider the optimization problem (P) defined by

$$(P) \quad \begin{array}{ll} \text{minimize} & c^Tx + \frac{1}{2}x^THx \\ \text{subject to} & x_j \in \{0, 1\}, \quad j = 1, \ldots, n, \end{array}$$

where H is an indefinite symmetric matrix. Problems of this type arise within combinatorial optimization, and the interest is to find a global minimizer.

One may compute lower bounds on the optimal value of (P) by considering relaxed problems.

(a) One way to relax (P) is to replace the constraints $x_j \in \{0,1\}$, $j = 1, \dots, n$, with $0 \le x_j \le 1$, $j = 1, \dots, n$. This gives a relaxed problem without discrete variables, according to

$$\begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2} x^T H x \\ \text{subject to} & 0 \le x_j \le 1, \quad j = 1, \dots, n, \end{array}$$

Explain way this relaxed problem is not very interesting in practise. .... (3p)

(b) An alternative way to create a relaxation to (P) is to introduce a symmetric matrix Y and formulate the semidefinite programming problem

$$(SDP) \qquad \begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2}\operatorname{trace}(HY) \\ \text{subject to} & \begin{pmatrix} Y & x \\ x^T & 1 \end{pmatrix} \succeq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ & Y = Y^T, \\ & y_{jj} = x_j, \quad j = 1, \dots, n. \end{array}$$

Show that if the constraint $Y = xx^T$ is added to (SDP), one obtains a problem which is equivalent to (P). .... (7p)

Hint: The following two results, which may be used without proof, might be useful:

(i) If H is an $n \times n$-matrix and x is an n-vector, then $\operatorname{trace}(Hxx^T) = x^T H x$.

(ii) If Y is a symmetric $n \times n$-matrix and x is an n-vector, then

$$\begin{pmatrix} Y & x \\ x^T & 1 \end{pmatrix} \succeq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{if and only if} \quad Y - xx^T \succeq 0.$$