

# Local Risk-Minimization for Defaultable Markets

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# Credit Risk

A **default risk** is a possibility that a counterparty in a financial contract will not fulfill a contractual commitment to meet her/his obligations stated in the contract.

If this happens, we say that the party defaults, or that a default event occurs.

More generally, by **credit risk** we mean the risk associated with any kind of credit-linked events, such as: changes in the credit quality (including downgrades or upgrades in credit ratings), variations of credit spreads and default events (bankruptcy, insolvency, missed payments).

**Defaultable Claims:** contingent agreements that are traded over-the-counter between default-prone parties. Each side of contract is exposed to the *counterparty risk* of the other party. *The underlying assets are assumed to be **insensitive** to credit risk.*

As for all the contingent claims, the most important problems to solve for the defaultable claims are the following ones:

- **pricing**

What price should the seller of a contingent claim  $H$  charge the buyer at time 0? (Contract's Valuation)

- **hedging**

How can the seller cover himself against the potential losses at time  $T$  (maturity) arising from a sale of  $H$ ?

# General Setting

- **Financial Market**

- *primary assets* on  $(\Omega, \mathcal{G}, \mathbb{P})$  :

- 1. *risky asset*  $S_t$

- 2. *money market account*  $B_t = \exp(\int_0^t r_s ds)$

- *default time*:  $\tau$

- $H_t = \mathbb{I}_{\{\tau \leq t\}}$  **default process**

- $W_t$  Brownian motion on  $(\Omega, \mathcal{G}, \mathbb{P})$

- $(\Omega, \mathcal{G}, \mathbb{P})$  endowed with the enlarged filtration

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$$

where  $\mathcal{F}_t = \sigma(W_u : u \leq t)$  and  $\mathcal{H}_t = \sigma(H_u : u \leq t)$

- **Hypothesis (H)**:  $W_t$  remains a (continuous) martingale (and then a Brownian motion) with respect to the enlarged filtration  $(\mathcal{G}_t)_{0 \leq t \leq T}$
- $\tau$  totally inaccessible  $\mathcal{G}_t$ -stopping time

- The **hazard process** under  $\mathbb{P}$

$$\Gamma_t = -\ln(1 - F_t), \quad \forall t \in [0, T]$$

where  $F_t = \mathbb{P}(\tau \leq t | \mathcal{F}_t)$  is the cumulative distribution function of  $\tau$ . We assume that there exists a non-negative integrable process  $\lambda_t$  (**hazard rate** or **intensity**) such that

$$\Gamma_t = \int_0^t \lambda_s ds, \quad \forall t \in [0, T]$$

- The **compensated process**

$$\hat{M}_t = H_t - \int_0^{t \wedge \tau} \lambda_u du = H_t - \int_0^t \tilde{\lambda}_u du,$$

with  $\tilde{\lambda}_t := \mathbb{I}_{\{\tau \geq t\}} \lambda_t$ , is a  $\mathcal{G}_t$ -martingale under  $\mathbb{P}$ .

- The risky asset dynamics is given by:

$$\begin{cases} dS_t &= \mu_t S_t dt + \sigma_t S_t dW_t \\ S_0 &= s_0, \quad s_0 \in \mathbb{R}_+ \end{cases}$$

where  $\sigma_t > 0$ ,  $\forall t \in [0, T]$  and  $\mu_t$ ,  $\sigma_t$ ,  $r_t$  are  $\mathcal{G}_t$ -adapted processes s.t.  
 $X_t = \frac{S_t}{B_t} \in L^2(\mathbb{P})$ ,  $\forall t \in [0, T]$ .

- We denote by

$$\theta_t = \frac{\mu_t - r_t}{\sigma_t}$$

the **market price of risk**. We assume that  $\mu$ ,  $\sigma$  and  $r$  are such that  $\mathcal{E} \left( - \int \theta dW \right)_T$  is square-integrable (N.A.).

- **Defaultable Claim**  $(\bar{X}, A, Z, \tilde{X}, \tau)$ , where
  - $\bar{X}$  is the **promised contingent claim**,
  - $A$  represents the **promised dividends**,
  - $Z$  is the **recovery process**,
  - $\tilde{X}$  is the **recovery claim**.
- In particular we assume  $A \equiv 0$ .
- Discounted payoff:

$$H = \frac{\bar{X}}{B_T} \mathbb{I}_{\{\tau > T\}} + \left( \frac{Z_\tau}{B_\tau} + \frac{\tilde{X}}{B_T} \right) \mathbb{I}_{\{\tau \leq T\}}$$

- The market extended with a defaultable claim is **not complete!**
- The process  $\hat{M}_t$  is **NOT** a tradeable asset.
- It makes sense to apply techniques used for pricing and hedging in incomplete markets.
- We choose **Quadratic Hedging Methods**.
- We apply the **Local Risk-Minimization** approach to defaultable markets.

## Local Risk-Minimization

## Local Risk-Minimization

**Problem:** we look for a hedging strategy  $\varphi$  with minimal cost which replicates the contingent claim  $H$ , i.e.  $\bar{V}_T(\varphi) = H$ .

- $X \in \mathcal{S}^2(\mathbb{P})$

$$X_t = X_0 + \int_0^t (\mu_s - r_s) X_s ds + \int_0^t \sigma_s X_s dW_s, \quad t \in [0, T]$$

- **(SC)**: the *mean-variance tradeoff*  $\hat{K}_t := \int_0^t \theta_s^2 ds$  is almost surely finite.
- We assume that  **$\hat{K}$  is uniformly bounded in  $t, \omega$** .

We denote by  $\Theta_s$  the space of  $\mathcal{G}$ -predictable processes  $\xi$  on  $\Omega$  such that

$$E \left[ \int_0^T (\xi_s)^2 \sigma_s^2 X_s^2 ds \right] + E \left[ \left( \int_0^T |\xi_s \mu_s X_s| ds \right)^2 \right] < \infty; \quad (1)$$

### Definition

- An  $L^2$ -strategy is a pair  $\varphi = (\xi, \eta)$  such that
  1.  $\xi \in \Theta_s$ ;
  2.  $\eta$  is a real-valued  $\mathcal{G}$ -adapted process such that the discounted portfolio value

$$\bar{V}_t(\varphi) = \xi_t \cdot X_t + \eta_t, \quad t \in [0, T]$$

is right-continuous and square-integrable.

- The **cost process** is defined by:

$$C_t = \bar{V}_t - \int_0^t \xi_s dX_s, \quad 0 \leq t \leq T.$$

- An  $L^2$ -strategy  $\varphi$  is called **mean-self-financing** if its cost process  $C_t(\varphi)$  is a  $\mathbb{P}$ -martingale.

### Definition

- Let  $H \in L^2(\mathcal{G}_T, \mathbb{P})$ . An  $L^2$ -strategy  $\varphi$  with  $\bar{V}_T(\varphi) = H$   $\mathbb{P}$ -a.e. is **pseudo-locally risk minimizing (plrm)** for  $H$  if  $\varphi$  is mean-self-financing and the martingale  $C(\varphi)$  is strongly orthogonal to the martingale part of  $X$ .

## Proposition

- A contingent claim  $H \in L^2(\mathbb{P})$  admits a plrm-strategy  $\varphi = (\xi, \eta)$  with  $\bar{V}_T(\varphi) = H$   $\mathbb{P}$ -a.s. if and only if  $H$  can be written as

$$H = H_0 + \int_0^T \xi_s^H dX_s + L_T^H \quad \mathbb{P} - \text{a.s.} \quad (2)$$

## Föllmer-Schweizer (FS) decomposition.

- **Plrm-strategy** (with respect to the discounted risky asset):  $\xi_t = \xi_t^H$ ,
- **Minimal cost:**  $C_t(\varphi) = H_0 + L_t^H$ .
- **Optimal discounted portfolio value:**  $\bar{V}_t(\varphi) = H_0 + \int_0^t \xi_s^H dX_s + L_t^H$  and  $\eta_t = \bar{V}_t(\varphi) - \xi_t^H X_t$ .

# The Minimal Martingale Measure

We see now how one can often obtain the FS decomposition by choosing a good martingale measure for  $X$ .

- A martingale measure  $\hat{\mathbb{P}}$  equivalent to  $\mathbb{P}$  with square-integrable density is called **minimal** if  $\hat{\mathbb{P}} \equiv \mathbb{P}$  on  $\mathcal{G}_0$  and if any square-integrable  $\mathbb{P}$ -local martingale which is strongly orthogonal to the martingale part of  $X$  under  $\mathbb{P}$  remains a local martingale under  $\hat{\mathbb{P}}$ .

It can be shown ( $\rightarrow$  **Föllmer-Schweizer**) that the following probability measure  $\hat{\mathbb{P}} \approx \mathbb{P}$

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} := \hat{Z}_T = \mathcal{E} \left( - \int \theta dW \right)_T \in L^2(\mathbb{P})$$

with  $\hat{Z} \in \mathcal{M}^2(\mathbb{P})$  and strictly positive on  $[0, T]$ , is the MMM for  $X$ .

## Theorem

Suppose  $X$  is continuous (and hence in our model satisfies (SC)). Consider the strictly positive local  $\mathbb{P}$ -martingale  $\widehat{Z} := \mathcal{E}(-\int \theta dW)$  and suppose that  $\widehat{Z} \in \mathcal{M}^2(\mathbb{P})$ .

Define the process  $\widehat{V}^H$  as follows

$$\widehat{V}_t^H := \widehat{E}[H|\mathcal{G}_t], \quad 0 \leq t \leq T.$$

Consider the GKW decomposition of  $\widehat{V}^H$  with respect to  $X$  under  $\widehat{\mathbb{P}}$

$$\widehat{V}_t^H = \widehat{E}[H|\mathcal{G}_t] = \widehat{V}_0^H + \int_0^t \widehat{\xi}_s^H dX_s + \widehat{L}_t^H. \quad (3)$$

If  $\widehat{\xi}^H \in \Theta_S$ ,  $\widehat{L}^H \in \mathcal{M}_0^2(\mathbb{P})$ , then (3) for  $t = T$  gives the FS decomposition of  $H$  and  $\widehat{\xi}^H$  gives a plrm-strategy for  $H$ . A sufficient condition to guarantee that  $\widehat{Z} \in \mathcal{M}^2(\mathbb{P})$  and the existence of a FS decomposition for  $H$  is that  $\widehat{K}$  is uniformly bounded.

# Local Risk-Minimization for Defaultable Markets

## Recovery Scheme at Maturity

The dynamics of the risky asset  $S_t$  may be influenced by the occurring of a default event and also the default time  $\tau$  itself may depend on the risky asset price behavior.

- $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t, \quad \forall t \in [0, T].$

- $H = \frac{\bar{X}}{B_T} I_{\{\tau > T\}} + h(\tau \wedge T) \frac{\bar{X}}{B_T} I_{\{\tau \leq T\}} = \frac{\bar{X}}{B_T} (1 + (h(\tau \wedge T) - 1)H_T),$   
where  $\bar{X}$  is  $\mathcal{G}_T$ -measurable.

- **Proposition.** For any  $\mathcal{G}_t$ -martingale  $N_t$  under  $\mathbb{P}$  we have

$$N_t = N_0 + \int_0^t \xi_u^N dW_u + \int_0^t \zeta_u^N d\hat{M}_u$$

where  $W$  and  $\hat{M}$  are strongly orthogonal.

- **Proposition.** The MMM  $\hat{\mathbb{P}}$  for  $X_t$  wrt  $(\mathcal{G}_t)_{0 \leq t \leq T}$  exists and its density is equal to  $\mathcal{E} \left( - \int \theta dW \right)_T$ .
- $\hat{M}_t$  is also a  $\hat{\mathbb{P}}$ -martingale.
- Hence for any  $\mathcal{G}_t$ -martingale  $N_t$  under  $\hat{\mathbb{P}}$  we have

$$\hat{N}_t = \hat{N}_0 + \int_0^t \xi_u^{\hat{N}} d\hat{W}_u + \int_0^t \zeta_u^{\hat{N}} d\hat{M}_u. \quad (4)$$

- **Problem:** find the FS decomposition for  $H$  by computing (4) for

$$\hat{V}_t^H = \hat{E}[H | \mathcal{G}_t] = \hat{E} \left[ \frac{\bar{X}}{B_T} (1 + (h(\tau \wedge T) - 1)H_T) \middle| \mathcal{G}_t \right].$$

## Solution

- $\hat{E} \left[ \frac{\bar{X}}{B_T} \middle| \mathcal{G}_t \right] = \hat{E} \left[ \frac{\bar{X}}{B_T} \right] + \int_0^t \bar{\xi}_s d\hat{W}_s + \int_0^t \bar{\eta}_s d\hat{M}_s$
- $\hat{E} \left[ \frac{\bar{X}}{B_T} (h(\tau \wedge T) - 1) H_T \middle| \mathcal{G}_t \right] = H_t (h(\tau \wedge T) - 1) \hat{E} \left[ \frac{\bar{X}}{B_T} \middle| \mathcal{F}_t \vee \mathcal{H}_T \right] +$   
 $(1 - H_t) \underbrace{e^{\int_0^t \lambda_s ds} \hat{E} \left[ \int_t^T \bar{Z}_s e^{-\int_0^s \lambda_u du} \lambda_s ds \middle| \mathcal{F}_t \right]}_{D_t},$

where  $\bar{Z}_t$  is an  $\mathcal{F}_t$ -predictable process such that

$$\bar{Z}_\tau = \hat{E} \left[ (h(\tau \wedge T) - 1) \frac{\bar{X}}{B_T} \middle| \mathcal{F}_{\tau-} \right].$$

- Then

- \*  $H_t(h(\tau \wedge T) - 1)\hat{E} \left[ \frac{\bar{X}}{B_T} \middle| \mathcal{F}_t \vee \mathcal{H}_T \right] = H_t \bar{Z}_\tau,$

- \*  $(1 - H_t)D_t = m_0 + \int_0^t \psi_s d\hat{W}_s - \int_0^t D_s d\hat{M}_s - \int_0^{t \wedge \tau} \bar{Z}_s \lambda_s ds.$

- The FS decomposition for  $H$  is given by:

$$\begin{aligned}
 H &= \hat{E} \left[ \frac{\bar{X}}{B_T} \right] + m_0 + \int_0^T \frac{1}{\sigma_s X_s} (\bar{\xi}_s + \mathbb{I}_{\{\tau \geq s\}} \xi_s^m e^{\int_0^s \lambda_u du}) dX_s \\
 &\quad + \int_0^T (\bar{Z}_s - D_s + \bar{\eta}_s) d\hat{M}_s,
 \end{aligned}$$

where

- $\bar{Z}_t$  is an  $\mathcal{F}_t$ -predictable process such that  $\bar{Z}_\tau = \hat{E} \left[ (h(\tau \wedge T) - 1) \frac{\bar{X}}{B_T} \middle| \mathcal{F}_{\tau-} \right]$ ,
- $m_t := \hat{E} \left[ \int_0^T \bar{Z}_s e^{-\int_0^s \lambda_u du} \lambda_s ds \middle| \mathcal{F}_t \right] = m_0 + \int_0^t \xi_s^m d\hat{W}_s$ ,
- $D_t := e^{\int_0^t \lambda_s ds} \hat{E} \left[ \int_t^T \bar{Z}_s e^{-\int_0^s \lambda_u du} \lambda_s ds \middle| \mathcal{F}_t \right]$ ,
- $\hat{E} \left[ \frac{\bar{X}}{B_T} \middle| \mathcal{G}_t \right] = \hat{E} \left[ \frac{\bar{X}}{B_T} \right] + \int_0^t \bar{\xi}_s d\hat{W}_s + \int_0^t \bar{\eta}_s d\hat{M}_s$ .

- The plrm-strategy is given by:

$$\xi_t^H = \frac{1}{\sigma_t X_t} \left( \bar{\xi}_t + \mathbb{I}_{\{\tau \geq t\}} \xi_t^m e^{\int_0^t \lambda_s ds} \right)$$

and the minimal cost is

$$C_t^H = \hat{E} \left[ \frac{\bar{X}}{B_T} \right] + m_0 + \int_0^t (\bar{Z}_s - D_s + \bar{\eta}_s) d\hat{M}_s.$$

## Example 1: $\tau$ dependent of $X$

- Corporate bond:  $H = 1 + (h(\tau \wedge T) - 1)H_T$  ( $\bar{X} = 1, B_t \equiv 1$ )

- $d\lambda_t = (b + \beta\lambda_t)dt + \alpha\sqrt{\lambda_t}d\hat{W}_t$ , with  $\lambda_0 = 0$  ( $\lambda$  is affine).

- $h(x) = \alpha_0\mathbb{I}_{\{x \leq T_0\}} + \alpha_1\mathbb{I}_{\{x > T_0\}}$ .

- The FS decomposition for  $H$  is given by:

$$\begin{aligned}
 H &= \alpha_0 + (\alpha_1 - \alpha_0)e^{-A(0, T_0)} + (1 - \alpha_1)e^{-A(0, T)} \\
 &\quad - \int_0^T \frac{1}{\sigma_s X_s} \left( (\alpha_1 - \alpha_0)\mathbb{I}_{\{s \leq T_0\}} e^{-A(s, T_0) - B(s, T_0)\lambda_s} B(s, T_0) \right. \\
 &\quad \left. + (1 - \alpha_1)e^{-A(s, T) - B(s, T)\lambda_s} B(s, T) \right) \sqrt{\lambda_s} dX_s + \int_0^T (h(s) - D_s - 1) d\hat{M}_s,
 \end{aligned}$$

where the functions  $A(t, T)$ ,  $B(t, T)$  satisfy the following equations:

$$\partial_t B(t, T) = \frac{\alpha^2}{2} B^2(t, T) - \beta B(t, T) - 1, \quad B(T, T) = 0 \quad (5)$$

$$\partial_t A(t, T) = -bB(t, T), \quad A(T, T) = 0, \quad (6)$$

that admit explicit solutions.

## Example 2: $X$ dependent on $\tau$

- $\mathcal{G}_t = \mathcal{F}_t \otimes \mathcal{H}_t$ ,  $\forall t \in [0, T]$  and  $dS_t = S_t[\mu_t(\tau)dt + \sigma_t(\tau)dW_t]$ , i.e. drift and volatility depend only on  $\tau$ , seen as an **exterior source of randomness**.
- **Proposition.** The plrm strategy  $\xi_t^H$  coincides with the predictable projection of the  $\tilde{\mathcal{G}}_t$ -predictable process  $\tilde{\xi}_t^H$  s.t.  $\int_0^T (\tilde{\xi}_s^H)^2 ds < \infty$  a.s. and

$$H = \hat{E} \left[ H \middle| \tilde{\mathcal{G}}_0 \right] + \int_0^T \tilde{\xi}_s^H d\hat{W}_s,$$

where  $\tilde{\mathcal{G}}_t = \mathcal{F}_t \otimes \mathcal{H}_T$ . (Case of *incomplete information*  $\rightarrow$  **Föllmer-Schweizer**).

- If  $\bar{X} = (S_T - K)^+$ , i.e.  $\bar{X}$  is given by the standard payoff of a call option, the plrm-strategy for  $H$  is given by:

$$\xi_t^H = E^{\hat{\mathbb{P}}, X} \left[ \mathbb{I}_{\{S_T \geq K\}} (1 + (h(\tau \wedge T) - 1)H_T) \middle| \mathcal{G}_{t-} \right].$$

### Example 3: Computation of $\bar{Z}_t$

- The introduction of the process  $\bar{Z}$  in may appear artificial. However it is necessary to find the FS decomposition.
- How can we compute  $\bar{Z}_t$ ?
- If  $\frac{\bar{X}}{B_T}$  is  $\mathcal{F}_T$ -measurable, then

$$\bar{Z}_t = (h(t \wedge T) - 1) (\hat{E} \left[ \frac{\bar{X}}{B_T} \right] + \int_0^t \bar{\xi}_s d\hat{W}_s).$$

- $\frac{\bar{X}}{B_T}$  (strictly)  $\mathcal{G}_T$ -measurable: Suppose that under  $\hat{\mathbb{P}}$ , the discounted asset price  $X_t$  is of the form

$$X_t = x_0 e^{\int_0^t \sigma(\tau \wedge s) d\widehat{W}_s - \frac{1}{2} \int_0^t \sigma(\tau \wedge s)^2 ds}, \quad x_0 > 0,$$

where  $\sigma$  is a sufficiently integrable positive Borel function, and  $\frac{\bar{X}}{B_T} = X_T^2$ .

- We obtain

$$\begin{aligned} \hat{E} \left[ \frac{\bar{X}}{B_T} \middle| \mathcal{F}_{\tau-} \right] &= \hat{E} \left[ x_0^2 e^{2 \int_0^T \sigma(\tau \wedge s) d\widehat{W}_s - \int_0^T \sigma(\tau \wedge s)^2 ds} \middle| \mathcal{G}_{\tau-} \right] \\ &= x_0^2 e^{\int_0^T \sigma(\tau \wedge s)^2 ds} \hat{E} \left[ e^{2 \int_0^T \sigma(\tau \wedge s) d\widehat{W}_s - 2 \int_0^T \sigma(\tau \wedge s)^2 ds} \middle| \mathcal{G}_{\tau-} \right] \\ &= x_0^2 e^{\sigma(\tau)^2 (T-\tau)} e^{2 \int_0^\tau \sigma(s) d\widehat{W}_s - \int_0^\tau \sigma(s)^2 ds} \end{aligned}$$

and

$$\bar{Z}_t = x_0^2 (h(t \wedge T) - 1) e^{\sigma(t)^2 (T-t)} e^{2 \int_0^t \sigma(s) d\widehat{W}_s - \int_0^t \sigma(s)^2 ds}$$

## Recovery Scheme at Default Time

A random recovery payment is received by the owner of the contract in case of default at time of default.

$$H = \frac{\bar{X}}{B_T} \mathbb{I}_{\{\tau > T\}} + \frac{Z_\tau}{B_\tau} \mathbb{I}_{\{\tau \leq T\}}$$

- $B_t$  is an  $\mathcal{F}_t$ -predictable process;
- $Z_t$  is a  $\mathcal{F}_t$ -predictable process such that  $\frac{Z_t}{B_t}$  is bounded;
- Since in our model we have a single default time, we assume that hedging stops after default.

## Local Risk-Minimization with $\mathcal{G}_t$ -strategies

The agent information takes into account the possibility of a default event.

### Definition

An  $L^2$ -strategy  $\varphi^{\mathcal{G}} = (\xi, \eta)$  is said a  $\mathcal{G}$ -plrm strategy if  $\xi_t \in \Theta_s$ ,  $\eta_t$  is  $\mathcal{G}_t$ -adapted and the discounted value process  $V_t(\varphi^{\mathcal{G}}) = \xi_t X_t + \eta_t$  is such that

$$V_t(\varphi^{\mathcal{G}}) = \int_0^t \xi_s dX_s + C_t(\varphi^{\mathcal{G}}), \quad t \in \llbracket 0, \tau \wedge T \rrbracket$$

where  $C_t$  is a square-integrable martingale strongly orthogonal to the martingale

part of  $X$  and  $V_{\tau \wedge T}(\varphi^{\mathcal{G}}) = H$ , i.e.

$$V_T(\varphi^{\mathcal{G}}) = \frac{\bar{X}}{B_T}, \quad \text{if } \tau > T, \quad V_{\tau}(\varphi^{\mathcal{G}}) = \frac{Z_{\tau}}{B_{\tau}}, \quad \text{if } \tau \leq T$$

## Proposition

Let  $G_t = \widehat{E}[H|\mathcal{G}_t]$ . There exists a pair of  $\mathcal{G}$ -predictable processes  $(\tilde{\xi}, \tilde{\zeta})$  s.t.

$$G_t = G_0 + \int_0^t \tilde{\xi}_s d\widehat{W}_s + \int_0^t \tilde{\zeta}_s d\widehat{M}_s.$$

- The FS decomposition for  $H$  is given by

$$\begin{aligned} H = & \widehat{E}[F] + m_0 + \int_0^T \mathbb{I}_{\{\tau \geq s\}} \left( \frac{e^{\int_0^s \lambda_u du} (\xi_s + \xi_s^m)}{\sigma_s X_s} \right) dX_s \\ & + \int_0^T \left( \mathbb{I}_{\{\tau \geq s\}} e^{\int_0^s \lambda_u du} \widehat{E}[F|\mathcal{F}_s] + \frac{Z_s}{B_s} - D_s \right) d\widehat{M}_s \end{aligned}$$

where

- $F = e^{-\int_0^T \lambda_s ds} \frac{\bar{X}}{B_T},$

- $m_t := \hat{E} \left[ \int_0^T \frac{Z_s}{B_s} e^{-\int_0^s \lambda_u du} \lambda_s ds \middle| \mathcal{F}_t \right] = m_0 + \int_0^t \xi_s^m d\hat{W}_s,$

- $\hat{E}[F | \mathcal{F}_t] = \hat{E}[F] + \int_0^t \xi_s d\hat{W}_s,$

- $D_t := e^{\int_0^t \lambda_s ds} \hat{E} \left[ \int_t^T \frac{Z_s}{B_s} e^{-\int_0^s \lambda_u du} \lambda_s ds \middle| \mathcal{F}_t \right].$

- The **plrm-strategy** is given by:

$$\xi_t^H = \mathbb{I}_{\{\tau \geq t\}} \frac{e^{\int_0^t \lambda_s ds} (\xi_t + \xi_t^m)}{\sigma_t X_t}$$

- The **minimal cost** is

$$C_t^H = \hat{E}[F] + m_0 + \int_0^t \left( \mathbb{I}_{\{\tau \geq s\}} e^{\int_0^s \lambda_u du} E^{\hat{\mathbb{P}}}[F | \mathcal{F}_s] + \frac{Z_s}{B_s} - D_s \right) d\hat{M}_s$$

$$\forall t \in [0, \tau \wedge T].$$

## Local Risk-Minimization with $\mathcal{F}_t$ -strategies

The agent obtains her information only by observing the non-defaultable assets.

- **Lemma.** For any  $\mathcal{G}_t$ -predictable process  $\tilde{\phi}_t$  there exists a  $\mathcal{F}_t$ -predictable process  $\phi_t$  such that

$$\mathbb{I}_{\{\tau \geq t\}} \tilde{\phi}_t = \mathbb{I}_{\{\tau \geq t\}} \phi_t, \quad t \in [0, T]$$

- There exists a  $\mathcal{F}_t$ -predictable process  $\tilde{X}_t$  (**pre-default discounted value** of  $X_t$ ) s.t.

$$\mathbb{I}_{\{\tau \geq t\}} \tilde{X}_t = \mathbb{I}_{\{\tau \geq t\}} X_t, \quad t \in [0, T]$$

- We can consider prices of primary non-defaultable assets stopped at  $\tau \wedge T$ .
- We can suppose that  $\mu$  and  $\sigma$  are  $\mathcal{F}$ -predictable.

- There do NOT exist  $\mathcal{F}$ -plrm strategies. We cannot hedge against the occurrence of a default by using only the information contained in  $\tilde{X}_t$ .
- We can think that the agent *invests* in  $X_t$  according to the information provided by the asset behavior *before default* and *adjusts* the portfolio value depending on the occurrence or not of the default.
- **Definition.** A strategy  $\varphi^{\mathcal{F}} = (\xi, C)$  is said a (pre)  $\mathcal{F}$ -plrm strategy if:
  1.  $\xi_t$  is a  $\mathcal{F}_t$ -predictable process satisfying (1);
  2.  $C_t$  is a  $\mathcal{G}_t$ -martingale strongly orthogonal to the martingale part of  $X$ ;
  3. the discounted value process  $V_t(\varphi^{\mathcal{F}}) = \xi_t X_t + \eta_t$  is such that

$$V_t(\varphi^{\mathcal{F}}) = \int_0^t \xi_s dX_s^\tau + C_t(\varphi^{\mathcal{F}}), \quad t \in \llbracket 0, \tau \wedge T \rrbracket$$

where

$$V_T(\varphi^{\mathcal{F}}) = \frac{\bar{X}}{B_T}, \text{ if } \tau > T, \quad V_\tau(\varphi^{\mathcal{F}}) = \frac{Z_\tau}{B_\tau}, \text{ if } \tau \leq T$$

- The **plrm-strategy** is given by:

$$\xi_t^H = \frac{e^{\int_0^t \lambda_s ds} (\xi_t + \xi_t^m)}{\tilde{\sigma}_t \tilde{X}_t}$$

- The **minimal cost** is

$$C_t^H = E^{\hat{\mathbb{P}}}[F] + m_0 + \int_0^t \left( e^{\int_0^s \lambda_u du} E^{\hat{\mathbb{P}}}[F | \mathcal{F}_s] + \frac{Z_s}{B_s} - D_s \right) d\hat{M}_s$$

$$\forall t \in [0, \tau \wedge T].$$

## Example

- Corporate bond:

$$H = \frac{1}{B_T}(1 - H_T) + \delta X_T H_T$$

where  $X_t = \hat{E} \left[ e^{-\int_0^T r_s ds} \middle| \mathcal{F}_t \right]$  and  $\frac{Z_t}{B_t} = \delta X_t$ ,  $t \in [0, T]$

- $dr_t = (b + \beta r_t)dt + \alpha \sqrt{r} d\hat{W}_t$ , with  $r_0 = 0$  ( $r$  is affine).
- $\lambda$  deterministic function
- The FS decomposition for  $H$  is given by:

$$H = e^{-A(0,T)} \left[ e^{-\int_0^T \lambda(u) du} + \delta(1 - e^{-\int_0^T \lambda(u) du}) \right] \\ - \int_0^T \mathbb{I}_{\{\tau \geq s\}} \frac{1}{\sigma_s X_s} \varphi_s dX_s + (\delta + 1) \int_0^T X_s e^{-\int_s^T \lambda(u) du} d\hat{M}_s,$$

where

$$\varphi_s = e^{-A(s,T) - B(s,T)r_s} \frac{B(s,T)}{B_s} \sqrt{r_s}$$

and the functions  $A(t, T)$ ,  $B(t, T)$  satisfy the following equations:

$$\partial_t B(t, T) = \frac{\alpha^2}{2} B^2(t, T) - \beta B(t, T) - 1, \quad B(T, T) = 0 \quad (7)$$

$$\partial_t A(t, T) = -bB(t, T), \quad A(T, T) = 0, \quad (8)$$

that admit explicit solutions.

- The **plrm-strategy** is given by:

$$\xi_t^H = -\frac{1}{\sigma_t X_t} \varphi_t$$

- The **minimal cost** is

$$C_t^H = e^{-\int_0^T \lambda(u) du - A(0,T)} + \delta e^{-A(0,T)} (1 - e^{-\int_0^T \lambda(u) du}) \\ + (\delta + 1) \int_0^T X_s e^{-\int_s^T \lambda(u) du} d\hat{M}_s$$

$$\forall t \in [0, \tau \wedge T].$$

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