Optimal strategies to sustain profitability of producing a commodity

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Investment projects producing a commodity

- Market price of the commodity: $X = (X_t)_{t \geq 0}$.

- Production activity (Two-regimes case):

Mode	On/open	off/closed	Default/def. closed
Indicator	1	0	†
Switching cost	a	D	$F(X_\gamma)$ (≤ 0)

- Running profit per unit time:

$$\Phi(t,x,u) = \begin{cases} \psi_1(t,x), & \text{if } u = 1; \text{ on/open,} \\ \psi_2(t,x), & \text{if } u = 0; \text{ off/closed.} \end{cases}$$

- Decision times: These are Stopping times.

$$
\begin{array}{ccccccccccc}
0 & \leq & \tau_1 \leq & \tau_2 \leq & \cdots \leq & \tau_n \leq & \cdots \leq & \gamma \leq & T \\
\downarrow & & \downarrow & \downarrow & \cdots & & & \downarrow & \\
\text{on} & & \text{off} & \text{on} & \cdots & & & \dagger &
\end{array}
$$

- State of the whole economic system:

$$
\begin{aligned}
(t, X_t, u_t) \quad & \text{if} \quad \tau_n < t \leq \tau_{n+1}, \\
(\gamma, X_\gamma) \quad & \text{if} \quad \text{in mode } \dagger \text{ (default)}.
\end{aligned}
$$

- The Management decision's strategy (admissible)

$$\delta = ((\tau_n)_{n\geq 1};\gamma)$$

such that $\tau_n \uparrow \gamma$.

$$
\begin{aligned}
(\tau_{2n}, \tau_{2n+1}] & \quad \text{in mode on/open,}\\
(\tau_{2n+1}, \tau_{2n+2}] & \quad \text{in mode off/closed,}\\
\gamma & \quad \text{in mode } \dagger \text{ (default).}
\end{aligned}
$$

- Expected profit using strategy δ:

$$
\begin{aligned}
J(\delta) = \ & E\left[\int_0^{\gamma} \Phi(s, X_s, u_s)ds\right]\\
& -E\left[\sum_{n\geq 1}\{a 1\!\!1_{[\tau_{2n-1}<\gamma]} + D 1\!\!1_{[\tau_{2n}<\gamma]}\}\right]\\
& +E\left[F(X_\gamma) 1\!\!1_{[\gamma<T]}\right].
\end{aligned}
$$

Optimal problem

Find the best strategies

$$\delta^* = ((\tau_n^*)_{n\geq 1}; \gamma^*)$$

for which

$$J(\delta^*) = \max_{\text{admissible } \delta} J(\delta).$$

- To sustain profitability only a finite number of decisions is required:

$$\max_{\text{admissible } \delta} J(\delta) = \max_{\textit{finite} \text{ admissible } \delta} J(\delta).$$

Brennan & Schwartz (85), Dixit (89),..., Zervos (03) discuss the case $T = \infty$ and need infinite number of decisions to sustain it.

Previous work

- Infinite horizon case ($T = \infty$) (PDE approach):

Brennan & Schawrtz (1985); Dixit(1989); Dixit & Pindyck (1994); Trigoergis (1993), (1996); Brekke & Øksendal (1991), (1994); Shirakawa (1997); Knudsen, Meister & Zervos (1999); Duckworth & Zervos (2001), (2001); Zervos (2003), Vath & Pham (2006).

- Finite horizon case ($T < \infty$) (Probabilistic approach):

1) Two-regimes case: Hamadène & Jeanblanc(2006); Hamadène & Hdhiri (2006); Porchet, Touzi & Warin (2006); Djehiche & Hamadène (2007).

2) Multiple-regimes case: Carmona & Ludkovski (2007); Djehiche, Hamadène & Popier (2007).

Main assumptions

- The filtration $\mathbf{F} = (\mathcal{F}_t)_{0 \le t \le T}$ is a Brownian filtration.

- The market price of the commodity $(X_t)_{t \ge 0}$ is

(i) $(\mathcal{F}_t)_t$–adapted and continuous;

(ii) $E[\sup_{0 \le t \le T} |X_t|^2] < \infty$.

- $F(x)$, $\psi_1(t,x)$, $\psi_2(t,x)$ are continuous.

- $|F(x)| + |\psi_1(t,x)| + |\psi_2(t,x)| \le C(1 + |x|)$ over $[0,T]$.

Probabilistic set up

• **The Snell Envelope**: The smallest supermartingale $(Z_t)_{0\leq t\leq T}$ that dominates a unif. integ. *rcll process* $(U_t)_{0\leq t\leq T}$ *is given by the following formula:*

$$Z_\theta = \textit{ess sup}_{\tau\geq\theta}\mathbb{E}[U_\tau|\mathcal{F}_\theta], \qquad Z_T = U_T,$$

τ and θ being stopping times. The process Z is continuous if U is continuous.

Moreover, if Z is continuous then:

$$\tau_\theta^* = \text{inf}\{s \geq \theta;\ Z_s = U_s\} \wedge T$$

is optimal and

$$Z_\theta = \mathbb{E}[Z_{\tau_\theta^*}|\mathcal{F}_\theta] = \mathbb{E}[U_{\tau_\theta^*}|\mathcal{F}_\theta].$$

A Verification Theorem

Consider the system of equations in (Y^1, Y^2):

$$\begin{cases} Y_t^1 = \text{ess sup}_{\tau \geq t} E[\int_t^\tau \psi_1(s, X_s)ds + (-a + Y_\tau^2) \vee F(X_\tau)1_{[\tau<T]}|\mathcal{F}_t], \\ \\ Y_t^2 = \text{ess sup}_{\tau \geq t} E[\int_t^\tau \psi_2(s, X_s)ds + (-D + Y_\tau^1) \vee F(X_\tau)1_{[\tau<T]}|\mathcal{F}_t]. \end{cases}$$

- Y_t^1 is the optimal expected profit, if at time t the production activity is in its open mode.

- Y_t^2 is the optimal expected profit, if at time t the production activity is in its closed mode.

- The system admits a unique solution (Y^1, Y^2), if it exits and is *continuous*.

- *Furthermore,*

$$Y_0^1 = \sup_{\delta \in \mathcal{D}} J(\delta).$$

Choosing the optimal strategy- the Snell Stopping times

Set $D_\tau(\zeta = \zeta') := \inf\{s \geq \tau, \zeta_s = \zeta'_s\} \wedge T$

and define the sequence of $\mathcal{F}$-stopping times $(\tau_n)_{n\geq 1}$ by:

$$\tau_1 = D_0(Y^1 = -a + Y^2) \wedge D_0(Y^1 = F(X))$$

$$\tau_2 = \begin{cases} D_{\tau_1}(Y^2 = -D + Y^1) \wedge D_{\tau_1}(Y^2 = F(X)) \text{ on } \{Y^1_{\tau_1} > F(X_{\tau_1})\} \\ \\ \tau_1, \quad \text{elsewhere,} \end{cases}$$

$$\tau_3 = \begin{cases} D_{\tau_2}(Y^1 = -a + Y^2) \wedge D_{\tau_2}(Y^1 = F(X)) \text{ on } \{Y^i_{\tau_i} > F(X_{\tau_i}), i = 1, 2\}, \\ \\ \tau_2, \quad \text{elsewhere,} \end{cases}$$

and for $n \geq 1$, set

$$L_n = \left\{ Y^1_{\tau_1} > F(X_{\tau_1}), Y^2_{\tau_2} > F(X_{\tau_2}), \ldots, Y^1_{\tau_{2n+1}} > F(X_{\tau_{2n+1}}) \right\}$$

$$\begin{aligned} \tilde{L}_n &= \left\{ Y^1_{\tau_1} > F(X_{\tau_1}), Y^2_{\tau_2} > F(X_{\tau_2}), \ldots, Y^1_{\tau_{2n+1}} > F(X_{\tau_{2n+1}}) \right\} \\ &\quad \cap \left\{ Y^2_{\tau_{2n+2}} > F(X_{\tau_{2n+2}}) \right\}, \end{aligned}$$

and

$$\tau_{2n+2} = \begin{cases} D_{\tau_{2n+1}}(Y^2 = -D + Y^1) \wedge \; D_{\tau_{2n+1}}(Y^2 = F(X)) \text{ on } L_n; \\ \\ \tau_{2n+1}, \qquad \text{elsewhere,} \end{cases}$$

$$\tau_{2n+3} = \begin{cases} D_{\tau_{2n+2}}(Y^1 = -a + Y^2) \wedge D_{\tau_{2n+2}}(Y^1 = F(X)) \text{ on } \tilde{L}_n; \\ \\ \tau_{2n+2}, \qquad \text{elsewhere.} \end{cases}$$

Finally, let $\gamma^* = \sup_{n \geq 1} \tau_n$.

- Then the strategy $\delta^* = ((\tau_n)_{n \geq 1}, \gamma^*)$ is optimal.

Existence of (Y^1, Y^2)

Main tool: Reflected Backward SDEs (El-Karoui *et al* (1997)).

Let $\xi \in L^2(\Omega, \mathcal{F}_T)$, $(f_t, S_t)_{t\leq T}$ sufficiently regular and such that $S_T \leq \xi$. Then, there exists a triple $(Y, Z, K) := (Y_t, Z_t, K_t)_{0\leq t\leq T}$ of sufficiently regular processes such that K_t is an increasing and continuous process with $K_0 = 0$ and

$$\begin{cases} Y_t = \xi + \int_t^T f_s\, ds - \int_t^T Z_s dB_s + K_T - K_t, \\ Y_t \geq S_t, \quad 0 \leq t \leq T, \\ \int_0^T (Y_t - S_t) dK_t = 0. \end{cases}$$

In addition, Y admits a Snell envelope representation:

$$Y_t = \text{ess sup}_{\tau \geq t} E[\int_t^\tau f_s ds + S_\tau 1\!\!1_{[\tau<T]} + \xi 1\!\!1_{[\tau=T]} | \mathcal{F}_t].$$

In our case, we have

$$\xi = 0, \qquad \text{(the terminal payoff)}$$

the barrier

$$S_t = \begin{cases} (-a + Y_t^2) \vee F(X_t), \\ or \\ (-D + Y_t^1) \vee F(X_t), \end{cases}$$

and the payoff per unit time

$$f_t = \begin{cases} \psi_1(t, X_t), \\ or \\ \psi_2(t, X_t). \end{cases}$$

Approximation scheme

Construct a sequence $(Y^{1,n}, Y^{2,n})_{n\geq 0}$ of reflected BSDEs that converges to (Y^1, Y^2) of the Verification Theorem.

$$Y_t^{1,0} = \text{ess sup}_{\tau\geq t} E[\int_t^\tau \psi_1(s, X_s)ds + F(X_\tau)1_{[\tau<T]}|\mathcal{F}_t],$$

$$\begin{cases} Y_t^{2,n} = \int_t^T \psi_2(s, X_s)ds - \int_t^T Z_s^{2,n} dB_s + K_T^{2,n} - K_t^{2,n}, \\ Y_t^{2,n} \geq (-D + Y_t^{1,n-1}) \vee F(X_t), \;\; 0 \leq t \leq T, \\ \int_0^T (Y_t^{2,n} - (-D + Y_t^{1,n-1}) \vee F(X_t))dK_t^{2,n} = 0. \end{cases}$$

and

$$\begin{cases} Y_t^{1,n} = \int_t^T \psi_1(s, X_s)ds - \int_t^T Z_s^{1,n} dB_s + K_T^{1,n} - K_t^{1,n}, \\ Y_t^{1,n} \geq (-a + Y_t^{2,n}) \vee F(X_t), \ \ 0 \leq t \leq T, \\ \int_0^T (Y_t^{1,n} - (-a + Y_t^{2,n}) \vee F(X_t)) dK_t^{1,n} = 0, \end{cases}$$

Properties of $(Y^{1,n}, Y^{2,n})$

$$\begin{cases} Y_t^{1,n} & = \text{ess sup}_{\tau \geq t} E[\int_t^\tau \psi_1(s, X_s)ds + (-a + Y_\tau^{2,n}) \vee F(X_\tau) 1\!\!1_{[\tau < T]} | \mathcal{F}_t] \\ Y_t^{2,n} & = \text{ess sup}_{\tau \geq t} E[\int_t^\tau \psi_2(s, X_s)ds + (-D + Y_\tau^{1,n-1}) \vee F(X_\tau) 1\!\!1_{[\tau < T]} | \mathcal{F}_t]. \end{cases}$$

• The sequences $(Y^{1,n})_n$ and $(Y^{2,n})_n$ are increasing (in n), bounded and converge to *rcll processes* $\tilde{Y}^1$ *and* $\tilde{Y}^2$,

where

$$\tilde{Y}_t^1 = \textit{ess sup}_{\tau \geq t} E[\int_t^\tau \psi_1(s, X_s)ds + (-a + \tilde{Y}_\tau^2) \vee F(X_\tau) 1\!\!1_{[\tau < T]} | \mathcal{F}_t]$$

$$\tilde{Y}_t^2 = \textit{ess sup}_{\tau \geq t} E[\int_t^\tau \psi_2(s, X_s)ds + (-D + \tilde{Y}_\tau^1) \vee F(X_\tau) 1\!\!1_{[\tau < T]} | \mathcal{F}_t].$$

• Using now reflected BSDEs with *rcll barriers we can show that* $\tilde{Y}^1$ *and* $\tilde{Y}^2$ *have no jumps. Therefore they are just* Y^1 *and* Y^2.

• *For* $i = 1, 2$, *we also have*

$$\lim_{n\to\infty} E[\sup_{0\le t\le T} |Y_s^{i,n} - Y_s^i|^2] = 0$$

and

$$P - a.s. \lim_{n\to\infty} \sup_{0\le t\le T} |Y_t^{i,n}(\omega) - Y_t^i(\omega)|^2 = 0.$$

Connection with variational inequalities

Consider the system of variational inequalities:

$$\begin{cases} \min\{v_1(t,x) - (-a + v_2(t,x)) \vee F(x), -\partial_t v_1(t,x) - Av_1(t,x) - \psi_1(t,x)\} = 0, \\ \min\{v_2(t,x) - (-D + v_1(t,x)) \vee F(x), -\partial_t v_2(t,x) - Av_2(t,x) - \psi_2(t,x)\} = 0, \\ v_1(T,x) = 0,\ v_2(T,x) = 0, \end{cases}$$

where,

$$A = \frac{1}{2}\sum_{i,j}(\sigma.\sigma^*)_{ij}(t,x)D_{ij} + \sum_i b_i(t,x)D_i.$$

is the infinitesimal generator of the Itô diffusion:

$$\begin{cases} dX_s^{tx} & = b(s, X_s^{tx})ds + \sigma(s, X_s^{tx})dB_s, \qquad t \le s \le T; \\ \\ X_s^{tx} & = x, \qquad \text{for } s \le t, \end{cases}$$

that models the market price of the commodities that influence the profitability.

Theorem Assume that there exist some positive constants C and $p \geq 1$ such that for $(t, x) \in [0, T] \times \mathbb{R}^k$

$$|\psi_1(t, x)| + |\psi_2(t, x)| + |F(x)| \leq C(1 + |x|^p).$$

Then there exist two deterministic functions $v^1(t, x)$ and $v^2(t, x)$ such that:

- v^1 and v^2 are continuous *in* (t, x) *and satisfy a polynomial growth condition.*

- *For each* $t \in [0, T]$ *and for any* $s \in [t, T]$,

$$Y_s^{1,tx} = v^1(s, X_s^{tx})$$

and

$$Y_s^{2,tx} = v^2(s, X_s^{tx}).$$

- *The pair* (v^1, v^2) *is a viscosity solution for the system of two variational inequalities.*

A special case

The case $F = -\infty$ corresponds to the default-free model discussed in Hamadène & Jeanblanc (2006). Let

$$Y = Y^1 - Y^2, \qquad \psi = \psi^1 - \psi^2, \qquad v = v^1 - v^2$$

• Y is a solution for a BSDE with two reflecting barriers D and $-a$

• the optimal strategy $(\tau_n)_n$ are those for which the process Y reaches successively the barriers D and $-a$

• in case when $X = X^{t,x}$ then $Y_s = v(s, X_s^{t,x})$ and v is a viscosity solution for the following double obstacle variational inequality:

$$\begin{cases} \min\{v(t,x) + a, \max\{-(\partial_t + A)v(t,x) - \psi(t,x), v(t,x) - D\}\} = 0, \\ V(T,x) = 0. \end{cases}$$

Extensions and limitations

• **Probabilistic setup.** The setup extends to the following cases:

a) The switching costs can be general positive continuous processes adapted to the Brownian filtration (with some integrability conditions) instead of constants.

b)The multiple switching regime case with general cost functions (Djehiche, Hamadène & Popier (2007)).

c) General impulse control problems with random coefficients (Djehiche, Hamadène & Hdiri (in progress))

- **Relation to viscosity solutions of variational inequalities**

a) We can only treat the case of switching costs that are deterministic and time dependent.

b) The setup extends to the multiple switching regime case with general cost functions (Djehiche, Hamadène & Popier (2007)). Carmona & Ludkovski (2007) provides a powerful numerical scheme.

Thank you for your attention!

Viscosity solutions

Let (v_1, v_2) be a pair of continuous functions on $[0, T] \times \mathbb{R}^k$ with values in $\mathbb{R}$ and such that $(v_1, v_2)(T, x) = 0$ for any $x \in \mathbb{R}^k$. The pair (v_1, v_2) is called

(i) A viscosity supersolution of the system if for any $(t_0, x_0) \in [0, T] \times \mathbb{R}^k$ and any pair of functions $(\varphi_1, \varphi_2) \in (C^{1,2}([0, T] \times \mathbb{R}^k))^2$ such that $(\varphi_1, \varphi_2)(t_0, x_0) = (v_1, v_2)(t_0, x_0)$ and (t_0, x_0) is a maximum of $\varphi_1 - v_1$ and $\varphi_2 - v_2$ respectively we have :

$$\begin{cases} \min\{v_1(t_0, x_0) - (-a + v_2(t_0, x_0)) \vee F(x_0), \\ \qquad -\partial_t \varphi_1(t_0, x_0) - A\varphi_1(t_0, x_0) - \psi_1(t_0, x_0)\} \geq 0 \end{cases}$$

and

$$\begin{cases} \min\{v_2(t_0, x_0) - (-D + v_1(t_0, x_0)) \vee F(x_0), \\ \qquad -\partial_t \varphi_2(t_0, x_0) - A\varphi_2(t_0, x_0) - \psi_2(t_0, x_0)\} \geq 0. \end{cases}$$

(ii) A viscosity subsolution of the system if for any $(t_0, x_0) \in [0, T] \times R^k$ and any pair of functions $(\varphi_1, \varphi_2) \in (C^{1,2}([0, T] \times \mathbb{R}^k))^2$ such that $(\varphi_1, \varphi_2)(t_0, x_0) = (v_1, v_2)(t_0, x_0)$ and (t_0, x_0) is a minimum of $\varphi_1 - v_1$ and $\varphi_2 - v_2$ respectively we have :

$$\left\{ \begin{array}{l} \min\{v_1(t_0, x_0) - (-a + v_2(t_0, x_0)) \vee F(x_0), \\ \qquad -\partial_t \varphi_1(t_0, x_0) - A\varphi_1(t_0, x_0) - \psi_1(t_0, x_0)\} \le 0 \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \min\{v_2(t_0, x_0) - (-D + v_1(t_0, x_0)) \vee F(x_0), \\ \qquad -\partial_t \varphi_2(t_0, x_0) - A\varphi_2(t_0, x_0) - \psi_2(t_0, x_0)\} \le 0. \end{array} \right.$$

(iii) A viscosity solution if it is both a viscosity supersolution and subsolution.