Solvability and Uniqueness for the Black-Scholes Equation in Stochastic Volatility Models

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A stochastic volatility model:

$$dX(t) = \sqrt{Y(t)}X(t)\,dW,$$

where the variance process Y is defined by

$$dY(t) = \beta(Y(t))\,dt + \sigma(Y(t))\,dV.$$

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Price of an option that pays $g(X(T))$ at T:

$$u(x,y,t) = E_{x,y,t}g(X(T))$$

The corresponding Black-Scholes equation is

$$\begin{cases} u_t + \frac{1}{2} y x^2 u_{xx} + \rho \sqrt{y} \sigma(y) x u_{xy} + \frac{1}{2} \sigma^2(y) u_{yy} + \beta(y) u_y = 0 \\ u(x, y, T) = g(x) \end{cases}$$

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The diffusion coefficient grows superquadratically, so neither existence or uniqueness of solutions to the equation is covered by the standard PDE-theory!

A Classical Result

The PDE

$$\begin{cases} u_t + \frac{1}{2} a_{11}(x,y) u_{xx} + a_{12}(x,y) u_{xy} + \frac{1}{2} a_{22}(x,y) u_{yy} = 0 \\ u(x,y,T) = g(x) \end{cases}$$

has a unique solution of at most polynomial growth if the operator is parabolic (i.e. $\sum_{i,j=1}^{2} a_{ij}(x,y)\xi_i\xi_j > 0$ for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$) and the coefficients a_{ij} are of at most quadratic growth:

$$|a_{ij}(x,y,t)| \leq C(1 + x^2 + y^2).$$

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- Review the classical PDE-theory for the Black-Scholes equation in one spatial dimension.

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- The Black-Scholes equation for stochastic volatility models (two spatial variables).

The Classical Set-Up

Stock price process:

$$dX(t) = \alpha(X(t), t)\, dW,$$

absorbed at 0. Assume that

$$|\alpha(x, t)| \leq C(1 + x).$$

Given a pay-off function g, define the option price by

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All Moments are Finite

Theorem

If $|\alpha(x,t)| \leq C(1+x)$, then all moments of $X(T)$ are finite.

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Bevis.

Ito $\Longrightarrow$

$$\begin{aligned} X^\gamma(T) &= x^\gamma + \gamma \int_0^T X^{\gamma-1}(s)\,\alpha(X(s),s)\,dW \\ &\quad + \frac{1}{2}\gamma(\gamma-1)\int_0^T X^{\gamma-2}(s)\alpha^2(X(s),s)\,ds. \end{aligned}$$

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Gronwall's lemma $\Longrightarrow$

$$EX^{\gamma}(T) \leq x^{\gamma}e^{CT}.$$

☐

Existence of Solutions

Corollary

Assume that $|\alpha(x,t)| \leq C(1+x)$ *and* $|g(x)| \leq C(1+x^N)$. *Then the function*

$$u(x,t) = E_{x,t}g(X(T))$$

solves the corresponding Black-Scholes equation.

Uniqueness of Solutions via the Maximum Principle

Theorem

Assume that $|\alpha(x,t)| \leq C(1+x)$. Then $u = 0$ is the only solution of at most polynomial growth to the equation

$$\begin{cases} u_t + \frac{1}{2}\alpha^2(x,t)u_{xx} = 0 \\ u(x,T) = u(0,t) = 0 \end{cases}$$

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For the proof one needs to find a supersolution $h(x,t)$ which grows faster than the candidate solution u as $x \to \infty$. Since u is of at most polynomial growth, say $|u(x,t)| \leq C(1+x^N)$, the function

$$h(x,t) = e^{Mt}(1+x^{N+1})$$

will do. h is indeed a supersolution if M is large enough:

$$h_t = Me^{Mt}(1+x^{N+1}) \quad \text{and} \quad \frac{1}{2}\alpha^2 h_{xx} \sim \alpha^2 x^{N-1} \sim x^{N+1}.$$

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Clearly, X is a non-negative local martingale, hence a supermartingale. Therefore

$$E_{x,t}X(T) \leq E_{x,t}X(t) = x.$$

It follows that $u(x,t) = E_{x,t}g(X(T))$ is finite if g is linear, so it solves the Black-Scholes equation.

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Models in which the stock price is a strict local martingale has been proposed to model bubbles, see Cox-Hobson (2005).

Non-Uniqueness of Solutions

If $E_{x,t}X(T) < x$, then $u^1 = E_{x,t}X(T)$ and $u^2(x,t) = x$ are two distinct solutions to the Black-Scholes equation

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If $u(x,t)$ is a solution to the Black-Scholes equation, then all functions of the form

$$u(x,t) + \lambda(x - E_{x,t}(X(T)))$$

are solutions as well.

A Stochastic Maximum Principle

Theorem

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Let v be a solution to the Black-Scholes PDE which is bounded from below. Define $Y(s) = v(X(s), s)$. By Ito's Lemma, $Y(s)$ is a local martingale. Since it is lower bounded, it is a supermartingale. Consequently,

$$v(x,t) = Y(t) \geq E_{x,t}Y(T) = E_{x,t}v(X(T),T) = E_{x,t}g(X(T)) = u(x,t).$$

□

Corollary

If g is bounded, then the Black-Scholes equation has a unique solution given by $E_{x,t}g(X(T))$.

A General Uniqueness Result

We also have the following stronger uniqueness result.

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Note that $h(x,t) = e^{Mt}(1+x)$ is a supersolution. Indeed,

$$h_t = Me^{Mt}(1+x) > 0 = \frac{1}{2}\alpha^2 h_{xx}.$$

Thus we have uniqueness in the class of functions which grow slower than x. $\square$

A Condition for X(t) to be a Strict Local Martingale

We have seen that uniqueness is lost in the class of linear functions if $x - E_{x,t}X(T) > 0$.

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Theorem

Assume that $|\alpha(x,t)| \geq x^{1+\delta}$ *for large* x*. If* $\delta > 0$*, then*

$$E_{x,t}X(T) = o(x^{\varepsilon}) \text{ as } x \to \infty$$

for any $\varepsilon > 0$*.*
If $\delta > 1/2$*, then* $E_{x,t}X(T)$ *is bounded in* x*.*

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If $\delta > 1/2$, then $E_{x,t}X(T)$ is bounded in x.

Corollary

If $|\alpha(x,t)| \geq x^{1+\delta}$ for large x, then $X(t)$ is a strict local martingale. Consequently, there are multiple solutions to the Black-Scholes equation (in the linear class).

Bevis.

It can be checked that the function

$$h(x,t) = e^{Mt}\frac{x}{1+t^n x^\beta}$$

is a supersolution if M, n and β are chosen appropriately. The result then follows from the Stochastic Maximum Principle. $\square$

Non-Existence of Solutions

Example

Assume that X follows the Constant Elasticity of Variance (CEV) model

$$dX = X^2\,dW.$$

Then X is a strict local martingale (actually, the first moment $E_{x,t}X(T)$ is bounded in x).

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For this model we have

- Uniqueness in the class of strictly sublinear functions.
- Existence in the class of functions of strictly sub-cubic growth.

Stochastic Volatility Models

Stock price:

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If Y reaches 0, then both X and Y remain constant.

Linear growth on the coefficients for Y:

$$|\beta(y,t)| \le C(1+y) \text{ and } |\sigma(y,t)| \le C(1+y)$$

Black-Scholes Equation for Stochastic Volatility Models

The option price is defined as

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$$u_t + \frac{1}{2} y \alpha^2(x,t) u_{xx} + \rho \sqrt{y} \sigma(y) \alpha(x,t) u_{xy} + \frac{1}{2} \sigma^2(y) u_{yy} + \beta(y) u_y = 0$$

with boundary conditions

$$\begin{cases} u(x,y,T) = g(x) \\ u(0,y,t) = g(0) \\ u(x,0,t) = g(x). \end{cases}$$

Existence and Uniqueness

As before, $X(t)$ is a lower bounded local martingale, hence a supermartingale. Therefore $E_{x,y,t}X(T) \leq x$, so $E_{x,y,t}g(X(T))$ is a solution to the BS-equation if g is at most linear.

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As for uniqueness, the function $h(x,t) = e^{Mt}(1 + x + y^m)$ is a supersolution (if M is large enough). Thus uniqueness always holds in the class of functions of strict sublinear growth in x and polynomial growth in y.

The Heston Model

In the Heston model,

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Results for the Heston Model

Theorem

Existence and uniqueness holds in the class of functions that are linear in x and polynomial in y.

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The function

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is a supersolution if M is large enough. ☐

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Corollary

In the Heston model, $X(t)$ is a true martingale.

The SABR Model

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If $\alpha < 1$, then all moments are known to be finite (Andersen-Piterbarg (2007)). Thus we have existence of solutions for any polynomial pay-off.

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References

- Andersen and Piterbarg: Moment explosions in stochastic volatility models. *Finance Stoch.* (2007).
- Cox and Hobson: Local martingales, bubbles and option prices. *Finance Stoch.* (2005).
- Sin. Complications with stochastic volatility models. *Adv. Appl. Probab.* (1998).

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Thank you for your attention!