No-arbitrage bounds on implied volatility

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The price of a European call option with strike K and maturity T

$$C_t(K,T) = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

- S_t is the price of the underlying stock
- r is the spot interest rate
- $$d_{1,2} = \frac{\log\left(\frac{S_t}{e^{-r(T-t)}K}\right)}{\sqrt{T-t}\sigma} \pm \frac{\sqrt{T-t}\sigma}{2},$$
- $\Phi(x) = \int_{-\infty}^{x}(2\pi)^{-1/2}e^{-t^2/2}dt$

- σ is the volatility of the underlying stock
- Unlike other parameters, not directly observed

$$\sigma^2 = \frac{\langle \log S \rangle_T}{T} = \frac{\mathrm{Var}(\log S_T)}{T}$$

- Liquid options priced by market already
- Options often quoted in terms of implied volatility

The assumptions

- No arbitrage
- Calls of all maturities and strikes liquidly traded
- Zero interest rate

Let $(S_t)_{t \geq 0}$ be a non-negative martingale with $S_0 = 1$.

Price of European call option with strike K and maturity T

$$C_t(K, T) = \mathbb{E}[(S_T - K)^+ | \mathcal{F}_t]$$

Define the function $F_{\mathrm{BS}} : \mathbb{R} \times \mathbb{R}_+ \to [0,1)$ via

$$F_{\mathrm{BS}}(k,v) = \begin{cases} \Phi\big(-\frac{k}{\sqrt{v}} + \frac{\sqrt{v}}{2}\big) - e^k \Phi\big(-\frac{k}{\sqrt{v}} - \frac{\sqrt{v}}{2}\big) & \text{if } v > 0 \\ (1-e^k)^+ & \text{if } v = 0 \end{cases}$$

Definition

The non-negative random variable $\Sigma_t(k,\tau)$ is defined by

$$\mathbb{E}[(\frac{S_{t+\tau}}{S_t} - e^k)^+ | \mathcal{F}_t] = F_{\mathrm{BS}}(k, \tau\, \Sigma_t(k,\tau)^2)$$

Question

How does the no-arbitrage assumption contrain the dynamics of the random field

$$(\Sigma_t(k,\tau))_{t\geq 0, k\in\mathbb{R}, \tau>0}?$$

Note the analogy with interest rate theory: If

$$(r_t)_{t \geq 0}$$

is the spot interest rate, the forward rate $f_t(\tau)$ is defined by

$$e^{-\int_0^\tau f_t(s)ds} = \mathbb{E}[e^{-\int_t^{t+\tau} r_s ds} | \mathcal{F}_t]$$

Note

$$f_t(0) = r_t$$

The Heath-Jarrow-Morton (1992) drift condition: If

$$df_t(\tau) = a_t(\tau)dt + b_t(\tau)dW_t$$

then

$$a_t(\tau) = \frac{\partial}{\partial \tau} f_t(\tau) + b_t(\tau) \int_0^\tau b_t(s)ds$$

A HJM approach to implied volatility: If

$$dS_t = S_t \sigma_t dW_t$$

for some process $(\sigma_t)_{t \geq 0}$ then

$$\Sigma_t(0,0) = \sigma_t$$

If

$$d\Sigma_t(k,\tau) = a_t(k,\tau)dt + b_t(k,\tau)dW_t$$

then

$$a_t(k,\tau) = \frac{\partial}{\partial \tau}\Sigma_t(k,\tau) + \text{ A MESS}$$

- Dupire (1994)
- Derman and Kani (1997)
- Schönbucher (1998)
- Cont and Da Fonseca (2001)
- Balland (2002)
- Berestycki,Busca, and Florent (2002, 2004)
- Durrleman (2004, 2007)
- Schwiezer and Wissel (2005, 2007)
- Jacod and Protter (2006)
- Carmona and Nadtochiy (2007)

Theorem (Dybvig–Ingersoll–Ross (1996))

Let

$$\limsup_{\tau \uparrow \infty} f_t(\tau) = \ell_t.$$

Then

$$\ell_t \geq \ell_s$$

for $t \geq s \geq 0$.

See Hubalek–Klein–Teichmann (2002) for a nice proof.

Parallel shifts of the term structure:

Proposition

Suppose

$$f_t(\tau) = f_0(\tau) + \xi_t$$

for some process $(\xi_t)_{t \geq 0}$. *If*

- $r_t \geq 0$ *almost surely*
- $\sup_{t \geq 0} \mathbb{E}(r_t) < \infty$

then $\xi_t = 0$ *for all* $t \geq 0$.

Parallel shifts of the implied volatility surface:

Conjecture (Ross)

Suppose

$$\Sigma_t(k,\tau) = \Sigma_0(k,\tau) + \xi_t$$

for some process $(\xi_t)_{t\geq 0}$. *Then* $\xi_t = 0$ *for all* $t \geq 0$.

Theorem (Lee (2004))

Let

$$\beta = \limsup_{k \uparrow \infty} \frac{\tau \Sigma(k, \tau)^2}{k}.$$

Then

$$\frac{1}{2\beta} + \frac{\beta}{8} + \frac{1}{2} = \sup\{p > 1 : \mathbb{E}(S_\tau^p) < \infty\}.$$

A similar formula holds as $k \downarrow -\infty$.

See Benaim and Friz (2006) for more precise asymptotics.

Proposition

$$\lim_{k \downarrow -\infty} \sqrt{\tau}\Sigma(k,\tau) - \sqrt{-2k} = \Phi^{-1}(\mathbb{P}(S_\tau = 0))$$

For example, if $dS_t = \alpha S_t^\beta dW_t$ with $0 < \beta < 1$ then

$$\sqrt{\tau}\Sigma(k,\tau) = \sqrt{-2k} - \Phi^{-1} \circ \Gamma(\frac{1}{2(1-\beta)}, \frac{1}{2(1-\beta)^2\alpha^2\tau}) + o(1)$$

where $\Gamma(\gamma, x) = \frac{1}{\Gamma(\gamma)} \int_0^x t^{\gamma-1} e^{-t} dt$.

Proposition

If $S_\tau > 0$ *a.s. then*

$$\Sigma(-k,\tau) = \hat{\Sigma}(k,\tau)$$

where

$$\mathbb{E}[(\hat{S}_\tau - e^k)^+] = F_{\mathrm{BS}}\big(k, \tau\, \hat{\Sigma}(k,\tau)^2\big)$$

and

$$\mathbb{P}(\hat{S}_\tau \le t) = \mathbb{E}(S_\tau \mathbb{1}_{\{S_\tau < t\}}).$$

For example, if

$$\begin{aligned} dS_t &= S_t\sigma(Y_t)dW_t \\ dY_t &= \alpha(Y_t)dt + \beta(Y_t)dW_t + \gamma(Y_t)dW_t^{\perp} \end{aligned}$$

then

$$\begin{aligned} d\hat{S}_t &= \hat{S}_t\sigma(\hat{Y}_t)dW_t \\ d\hat{Y}_t &= (\alpha(\hat{Y}_t) + \beta(\hat{Y}_t)\sigma(\hat{Y}_t)^2)dt + \beta(\hat{Y}_t)dW_t + \gamma(\hat{Y}_t)dW_t^{\perp}. \end{aligned}$$

Renault-Touzi's (1996) result follows from this.

Proposition

The smile is symmetric

$$\Sigma(k, \tau) = \Sigma(-k, \tau)$$

if and only if

$$\mathbb{E}(S_\tau^p) = \mathbb{E}(S_\tau^{1-p})$$

for all $0 \leq p \leq 1$*.*

Theorem

Assume that the law of S_τ is continuous for all $\tau > 0$.

- $$\partial_k \Sigma(k,\tau)^2 < \frac{4}{\tau} \text{ for all } k \geq 0$$
  $$\partial_k \Sigma(k,\tau)^2 > -\frac{4}{\tau} \text{ for all } k \leq 0$$
- *If $S_t \to 0$ a.s., then*
  $$\limsup_{\tau \uparrow \infty} \sup_{k \in [-M,M]} \tau |\partial_k \Sigma(k,\tau)^2| \leq 4$$

This sharpens the result of Carr and Wu (2002).

The inequality is sharp in the sense that there exists a martingale $(S_t)_{t\geq 0}$ such that

$$\tau \partial_k \Sigma(k,\tau)^2 \to -4$$

as $\tau \uparrow \infty$ uniformly for $k \in [-M, M]$.

For comparison, a consequence of Lee's results:

Proposition

Assume that the law of S_τ is continuous for all $\tau > 0$.

- *There exists a $k_+ \geq 0$ such that*

$$\partial_k \Sigma(k,\tau)^2 < \frac{2}{\tau} \text{ for all } k \geq k_+$$

- *If $S_\tau > 0$ a.s. then there exists a $k_- \leq 0$ such that*

$$\partial_k \Sigma(k,\tau)^2 > -\frac{2}{\tau} \text{ for all } k \leq k_-$$

The condition $S_t \to 0$ almost surely is natural since the following are equivalent:

- $S_t \to 0$ almost surely.
- There exists a $k \in \mathbb{R}$ such that $\tau\Sigma(k,\tau)^2 \uparrow \infty$
- $\tau\Sigma(k,\tau)^2 \uparrow \infty$ for all $k \in \mathbb{R}$.
- $\mathbb{E}[(S_\tau - K)^+] \uparrow S_0$ for all $K > 0$.
- There exists a $K > 0$ such that $\mathbb{E}[(S_\tau - K)^+] \uparrow S_0$

Proposition

$$\lim_{\tau \uparrow \infty} \Sigma(k, \tau) - \left(-8 \frac{\log \mathbb{E}(S_\tau \wedge 1)}{\tau} \right)^{1/2} = 0$$

for all $k \in \mathbb{R}$.

See Lewis (2000) and Jacquier (2006, 2007) for detailed $T \uparrow \infty$ asymptotics for some popular models.

Theorem

For any $k_1, k_2 \in \mathbb{R}$ *we have*

$$\limsup_{\tau \uparrow \infty} \Sigma_t(k_1, \tau) - \Sigma_s(k_2, \tau) \geq 0$$

for $t \geq s \geq 0$. *There exist examples for which the inequality is strict.*

Theorem

Suppose

$$\Sigma_t(k,\tau) = \Sigma_0(k,\tau) + \xi_t$$

for some process $(\xi_t)_{t\geq 0}$. *Then* $\xi_t \geq 0$.
Furthermore, let

$$g_p(t) = \frac{1}{p(p-1)} \log \mathbb{E}(S_t^p).$$

If there exists a $p \in \mathbb{R}$ *and a* $\tau > 0$ *such that*

$$g_p(t+\tau) \leq g_p(t) + g_p(\tau)$$

then $\xi_t = 0$.

If $\xi_t = 0$ for all $t \geq 0$ and $S_t \to 1$ in probability as $t \downarrow 0$ then $(S_t)_{t \geq 0}$ is an exponential Levy process.

The conjecture is false for implied average variance $\Sigma_t(k,\tau)^2$.

Proposition

The martingale

$$S_t = e^{-t^4/2 + W_{t^2}}$$

has the property that

$$\Sigma_t(k,\tau)^2 = \Sigma_0(k,\tau)^2 + \xi_t$$

where

$$\Sigma_0(k,\tau) = \sqrt{\tau}$$

and

$$\xi_t = 2t$$

Proposition

Suppose

$$\Sigma_t(k,\tau) = \Sigma_0(k,\tau) + \xi_t$$

for some process $(\xi_t)_{t\geq 0}$. *If*

$$S_t = e^{-F(t)/2 + W_{F(t)}}$$

for a positive increasing some function F *with* $F(0) = 0$. *Then*

$$\xi_t = 0.$$