Convexity in Interest rate Theory

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Convexity results of E. Ekström and J. Tysk

N-dimesional considerations

A result for N-dimensional Ito-diffusions

Consider a one-dimensional short rate model

$$dr_t = \beta(r_t)dt + \sigma(r_t)dW_t$$

on a filtered stochastic basis $(\Omega, \mathcal{F}, P)$ carrying a one-dimensional Brownian motion W.
Under suitable assumptions on β, σ we can guarantee that

$$G(r,t) := E(\exp(-\int_0^t r_s ds)g(r_t))$$

is $C^{2,1}$-regular for sufficiently regular pay-off functions g.

The Theorem of Ekström/Tysk

Assume sufficient regularity on β, σ, g and assume that g is a **non-negative, convex, decreasing pay-off**. If

$$\beta_{rr}(r) \leq 2$$

for all r, then $r \mapsto G(r,t)$ is convex in r.
J. Ekström and J. Tysk also prove assertions on monotonicity (with respect to changes in volatility and drift), log-convexity and log-concavity..

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- How does a multi-dimensional version of the previous theorem look like?
- Does it hold for jump-diffusions or Lévy-driven processes, too?

Fix $N, d \geq 1$. We consider a stochastic differential equation

$$dr_s = \beta(r_s)ds + \sum_{i=1}^{d} \sigma_i(r_s)dW_s^i$$

in $\mathbb{R}^N$ and a non-zero linear functional l on $\mathbb{R}^N$. We assume the coefficients to satisfy C^∞-boundedness conditions and the initial value r to be deterministic. We **do not assume** ellipticity.

An approach by Semi-group Theory

Consider a Banach space X of pay-off functions and apply criteria for invariant sets in this situation.

- A semigroup P leaves C invariant, i.e. $P_t C \subset C$ for all $t \geq 0$, if and only if $E(P_\tau x) \in C$ for all bounded random times τ and all $x \in C$ (Resolvent criterion).

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- Let $C \subset X$ be a closed convex set and assume that for some $n \geq 0$ the set $C_n := D(A^n) \cap C$ has non-empty interior in the space $D(A^n)$ and has closure C in X. A semigroup P leaves C invariant, i.e. $P_t C \subset C$ for all $t \geq 0$ if and only if $l(Ax) \leq 0$ for all $x \in D(A^{n+1} \cap \partial_n C_n$ and some supporting hyperplane l at x.

Considering the general situation this boils down to checking in an appropriate Banach space of functions an inequality for boundary points of a convex set of functions.
Take $N = d = 1$ and consider the closed convex set of **convex, non-negative and decreasing pay-off functions**. The crucial boundary points in some power domain are those, where the second derivative vanishes and one can check easily the equivalence of the semi-group criterion to the Ekström/Tysk condition.

A Theorem for Jump Diffusions

Consider a diffusion r leaving a certain convex set C of functions invariant, then jump diffusion extensions leave the same convex set of functions C invariant.

In the particular previous case we obtain for

$$dr_t = \beta(r_t)dt + \sigma(r_t)dW_t + \delta(r_t)dN_t$$

that $r \mapsto G(r,t)$ is convex, decreasing and positive if

$$\beta_{rr} \leq 2$$

and $r \mapsto r + \delta(r)$ is linear and increasing.

An analytic approach

Let $g : \mathbb{R}^N \to \mathbb{R}$ be a **convex, non-negative function** with sufficient regularity such that the following calculations make sense. We consider

$$G(r,t) = E(exp(-\int_0^t l(r_s)ds)g(r_t))$$

for $t \geq 0$.

First and Second Variation

We denote the derivatives with respect to r by T (tangent map). We shall apply the first and second variation of the stochastic process r with respect to its initial value in a deterministic direction v, which is a stochastic process again,

$$\partial_v r_s = T\beta(r_s)\partial_v r_s ds + \sum_{i=1}^{d} T\sigma_i(r_s)\partial_v r_s dW_s^i, \quad \partial_v r_0 = v$$

and

$$\begin{aligned}\partial_v^2 r_s = {} & \big(T\beta(r_s)\partial_v^2 r_s + T^2\beta(r_s)(\partial_v r_s)^2\big)ds + \\ & + \sum_{i=1}^{d}\big(T\sigma_i(r_s)\partial_v^2 r_s + T^2\sigma_i(r_s)(\partial_v r_s)^2\big)dW_s^i\end{aligned}$$

with $\partial_v^2 r_0 = 0$.

Taking derivatives

The first derivative of G with respect to r,

$$TG(r,t)v = -E(exp(-\int_0^t l(r_s)ds)\int_0^t l(\partial_v r_s)ds\; g(r_t))+ \\ + E(exp(-\int_0^t l(r_s)ds)Tg(r_t)\partial_v r_t)$$

and...

Taking derivatives

...the second one

$$\begin{aligned} T^2G(r,t)(v)^2 = {} & E(exp(-\int_0^t l(r_s)ds)(\int_0^t l(\partial_v r_s)ds)^2 g(r_t))- \\ & - E(exp(-\int_0^t l(r_s)ds)\int_0^t l(\partial_v^2 r_s)ds\; g(r_t))- \\ & - E(exp(-\int_0^t l(r_s)ds)\int_0^t l(\partial_v r_s)ds\; (Tg(r_t)\partial_v r_t))- \\ & - E(exp(-\int_0^t l(r_s)ds)\int_0^t l(\partial_v r_s)ds\; (Tg(r_t)\partial_v r_t))+ \\ & + E(exp(-\int_0^t l(r_s)ds)(T^2g(r_t)(\partial_v r_t)^2))+ \\ & + E(exp(-\int_0^t l(r_s)ds)(Tg(r_t)\partial_v^2 r_t)). \end{aligned}$$

There is a symmetric structure, which is best revealed by the following process

$$C_t = 2\int_0^t l(\partial_v r_s)ds\,\partial_v r_t - \partial_v^2 r_t.$$

Hence we can write that

$$\begin{aligned} T^2 G(r,t)(v)^2 =& E(exp(-\int_0^t l(r_s)ds)g(r_t)\int_0^t l(C_s)ds) - \\ &- E(exp(-\int_0^t l(r_s)ds)Tg(r_t)C_t) + \\ &+ E(exp(-\int_0^t l(r_s)ds)(T^2 g(r_t)(\partial_v r_t)^2)) \end{aligned}$$

by gathering the respective terms and applying that

$$\int_0^t l(C_s)ds = (\int_0^t l(\partial_v r_s)ds)^2 - \int_0^t l(\partial_v^2 r_s)ds.$$

The Theorem

Assume first that $l(\partial_v r_t) \geq 0$ and $Tg(r_t)\partial_v r_t \leq 0$ almost surely for all $t \geq 0$, then we obtain that

$$TG(r,t)v \leq 0$$

for $r \in \mathbb{R}^N$ and $t \geq 0$. Next we assume (without granting the first assumption) that

$$2l(v)(TG(r,t)v) - TG(r,t)T^2\beta(r)(v)^2) > 0$$

for all $v, r \in \mathbb{R}^N$ with $l(v) > 0$ and $t \geq 0$. Then we conclude that $r \mapsto G(r,t)$ is convex.

Corollary

- Fix $N, d \geq 1$. Assume that the drift β is linear and that $g = 1$. If $l(\partial_v r_t) \geq 0$ for all $v \in \mathbb{R}^N$ with $l(v) \geq 0$, then
$$r \mapsto E(\exp(\int_0^t l(r_s) ds))$$
is convex with respect to the initial values.

Corollary

- Fix $N, d \geq 1$. Assume that the drift β is linear and that $g = 1$. If $l(\partial_v r_t) \geq 0$ for all $v \in \mathbb{R}^N$ with $l(v) \geq 0$, then
$$r \mapsto E(\exp(\int_0^t l(r_s)ds))$$
is convex with respect to the initial values.
- Assume that $N = 1$, $d \geq 1$ and that g is decreasing. We choose $l(x) = x$ and $v = 1$. Since $\partial_v r_0 = v$ we have that $l(\partial_v r_t) \geq 0$ by continuity of the trajectories and the fact that $\partial_v r_t \neq 0$ almost surely for $t \geq 0$. Consequently we obtain in this case $TG(r,t)v \leq 0$. Supposing finally that
$$2 - \beta_{rr}(r) \geq 0$$
yields convexity such as in Ekström/Tysk.

The Proof

We consider the $5N$-dimensional Markov process X composed of $(r_t, \partial_v r_t, \partial_v^2 r_t, \int_0^t r_s ds, \int_0^t \partial_v r_s ds)$ for the approriate initial value $(r, v, 0, 0, 0)$. We interpret the previous expectations as solutions of the respective heat equation for a particular initial function F. We know from Dynkin's formula that

$$E(F(X_t)) = F(X_0) + \int_0^t E(\mathcal{A}F(X_s))ds,$$

where $\mathcal{A}$ denotes the generator of X.

The Proof

We choose

$$F(x_1, x_2, x_3, x_4, x_5) = \exp(-l(x_4))\, Tg(x_1)(2l(x_5)x_2) - x_3),$$

which yields

$$\begin{aligned} \mathcal{A}F(x_1, x_2, x_3, x_4, x_5) = & -l(x_1)\exp(-l(x_4))\, Tg(x_1)(2l(x_5)x_2) - x_3) + \\ & + \exp(-l(x_4))\, Tg(x_1)(2l(x_5)x_2) - x_3) + \\ & + \exp(-l(x_4))\, Tg(x_1) \times \\ & \times (2l(x_2)x_2 + 2l(x_5)\, T\beta(x_1)x_2 - T^2\beta(x_1)(x_2)^2 - T\beta(x_1)x_3). \end{aligned}$$

The Proof

We assume that $T^2G(r,t)(v)^2 = 0$ and want to prove that the derivative at t of this quantity has to be positive. We obtain by the previous consideration and $g = G(r,t)$

$$\begin{aligned}\frac{\partial}{\partial t}|_{t=0} T^2G(r,t)(v)^2 &\geq -\mathcal{A}F(r,v,0,0,0) \\ &= -(2l(v)TG(r,t)v - TG(r,t)T^2\beta(r)(v)^2) > 0.\end{aligned}$$

Discussion

For the proof we need the following structures:

- A Markov process admitting sufficiently regular solutions of its associated heat equation.

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- A Markov process admitting first and second variations (see for instance Ph. Protter's book for conditions when semi-martingale-driven SDEs admit stochastic flows).

Discussion

For the proof we need the following structures:

- A Markov process admitting sufficiently regular solutions of its associated heat equation.
- A Markov process admitting first and second variations (see for instance Ph. Protter's book for conditions when semi-martingale-driven SDEs admit stochastic flows).
- The sign of the generator $\mathcal{A}$ on

  $$F(x_1, x_2, x_3, x_4, x_5) = \exp(-I(x_4))\, Tg(x_1)(2I(x_5)x_2) - x_3)$$

  at $(r, v, 0, 0, 0)$ is decisive.

- Erik Ekström, Johan Tysk, *Convexity Theory for the Term structure equation*, preprint (2007).
- Erik Ekström, Johan Tysk, *Properties of option prices in models with jumps*. to appear in Math. Finance (2007).

- Thank you for your attention!

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- Further informations on this research @ *Publications* soon on my webpage (google Josef Teichmann).