

Local Variance Gamma Option Pricing Model

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April 28, 2009

Automated Option Market Making

- ▶ As markets move from an open outcry system to computerized exchanges, the question arises as to how to automate the pricing and risk management of derivative securities.
- ▶ We take the view that this problem is presently unsolved and this explains why firms continue to use traders to price and hedge.
- ▶ The true “revenge of the nerds” will begin when a quant develops a solution to this problem.

First Statement of The Problem

- ▶ Banks, software vendors, and other options market participants have historically hired armies of PhD's to tackle the following apparently simple problem.
- ▶ **Problem Statement:** Given the current market price of the underlying asset and also given market option quotes at several given strikes and terms, provide options quotes at any strike and term in a specified set.
- ▶ For simplicity, we suppose that both the supplied quotes and the produced quotes are mid-points of the bid and the ask.
- ▶ For simplicity, we suppose that both the input quotes and the output quotes are for European options.
- ▶ We will refer to the given strikes, terms, and mids as “listed”. The problem of inferring option prices at non-listed strikes and terms arises on both exchanges and OTC.

How Many PhD's Does it Take to Connect the Dots?

- ▶ Since the supplied mids can be depicted as points in the first octant of \mathbb{R}^3 , it first appears that armies of PhD's are merely being asked to connect the dots.
- ▶ As my 7 year old daughter mastered this intellectual challenge several years ago, one has to wonder if this is not the financial equivalent of the lightbulb joke.
- ▶ It is my contention that once all the desired properties of the solution are clarified, no one has ever published a solution that successfully connects the dots.

Glorified Interpolation?

- ▶ Recall that the option market-makers' pricing problem only asks that option prices be provided for strikes and terms in a specified set.
- ▶ When this set lies in the convex hull of listed strikes and terms, the option market makers pricing problem can be properly described as “glorified interpolation” .
- ▶ When at least part of the desired set lies beyond listed strikes and terms, extrapolation is involved. Philosophically, there is no unique solution to any extrapolation problem.
- ▶ What makes this problem even more difficult is that there is a long laundry list of desirable properties that a solution is asked to possess. Unfortunately, these properties are typically only discovered after much effort has been spent solving the wrong problem.

A Partial List of Desirable Properties

- ▶ The following highly demanding criteria can usually be expressed in any of several mathematically equivalent spaces. Common spaces used for the output include option price space, implied vol space, and local vol space. For simplicity, we treat the independent variables as strike K and term T .
- ▶ robust to input data, i.e. desirable output is produced no matter what is input.
- ▶ well-posedness, i.e. small changes in inputs lead only to small changes in output.
- ▶ locality, i.e. a small change in one input has only a local effect on the output.
- ▶ ability to accept related inputs eg. American options, variance swaps, CDS, options on other underlyings.

A Partial List of Desirable Properties (Con'd)

- ▶ no model-free arbitrages in the output (assuming wide support for possible levels of the underlying uncertainty).
- ▶ exact price consistency with given mids (implied by above).
- ▶ smooth output (except for option prices at $T = 0$).
- ▶ perfect out-of-sample prediction in K, T space.
- ▶ ability to extrapolate in S, t space i.e., can produce deltas, gammas, thetas, spot slides, and time slides.
- ▶ resonance with financial orthodoxy eg. nonnegative gammas.

A Partial List of Desirable Properties (Con'd)

- ▶ perfect out-of-sample prediction in the Cartesian product of spot prices S and calendar time t .
- ▶ internal consistency eg. quantities assumed constant in a parametric model should not vary with S , t or other observables.
- ▶ real-time computational speed.
- ▶ numerical robustness (always works on finite precision computers).

A Partial List of Desirable Properties (Con'd)

- ▶ ability to uniquely and accurately price related derivatives such as American options, variance swaps, barrier options, and other exotics.
- ▶ parsimony
- ▶ distinct economic role for parameters that permit identification.
- ▶ implied \mathbb{Q} dynamics not ridiculously far from observable \mathbb{P} dynamics - no stat. arb.

Origin and Motivation

- ▶ Local Variance Gamma (henceforth LVG) is a work in progress which satisfies many, but not all, of the criteria just mentioned. The model takes its name from the fact that it simultaneously generalizes both Dupire's 1994 Local Variance model and Madan and Seneta's 1990 Variance Gamma model.
- ▶ The use of the word "local" in the LVG name is apt for two reasons:
 1. As in local vol models, the uncertainty in stock returns is assumed to depend (only) on the stock price level and time.
 2. Calibration is local, if N option prices are given, one does N univariate searches, rather than one global search for N parameters.

Inputs and Outputs of LVG

- ▶ The input into LVG is any set of arbitrage-free option prices at discrete strikes and maturities on a regular grid. The output is an exactly-consistent arbitrage-free price process for the underlying asset.
- ▶ Others have also done this; what is special about LVG AFAIK is that calibration to discrete strike and maturity option prices is local and yet the produced dynamics for the underlying asset price evolve in cont. time with a cont. state space.

Dynamical Description in Words

- ▶ In LVG, the underlying forward price process is assumed to be a driftless diffusion running on an independent gamma clock. The resulting continuous-time price process is a pure jump Markov martingale with infinite activity.
- ▶ The LVG price process is not a Lévy process, but it enjoys time & space homogeneity between listed strikes & terms.
- ▶ If option prices at just 1 term are given, the process is time-homogeneous.
- ▶ Likewise, if option prices at just 1 strike per term are given, the process is space-homogeneous (given by time-dependent VG).
- ▶ When 2 or more option prices at 1 term or strike are given, the diffusion coefficient is assumed to be piecewise constant, i.e, it jumps at listed strikes and maturities, but is otherwise flat.

Assumptions

- ▶ In this presentation, I will just show the case where option prices at a single term are given.
- ▶ Furthermore, interest rates and dividends are zero and limited liability is not respected. (all of this can be relaxed).

Dynamical Description in Symbols

- ▶ Let D be a time homogeneous diffusion starting at S_0 :

$$dD_s = a(D_s)dW_s, \quad s \geq 0,$$

where the diffusion coefficient $a(x) : \mathbb{R} \mapsto \mathbb{R}$ is initially unspecified.

- ▶ Let $\{\Gamma_t, t > 0\}$ be an independent gamma process with Lévy density $k_\Gamma(t) = \frac{e^{-\alpha t}}{t^* t}, t > 0$ for parameters $t^* > 0$ and $\alpha > 0$.
- ▶ The marginal distribution of a gamma process at time $t \geq 0$ is a gamma distribution with PDF:

$$\mathbb{Q}\{\Gamma_t \in ds\} = \frac{\alpha^{t/t^*}}{\Gamma(t/t^*)} s^{t/t^*-1} e^{-\alpha s}, s > 0, \alpha > 0, t^* > 0.$$

Specifying the Gamma Process

- ▶ Recall that the marginal distribution of a gamma process at time $t \geq 0$ is a gamma distribution with PDF:

$$\mathbb{Q}\{\Gamma_t \in ds\} = \frac{\alpha^{t/t^*}}{\Gamma(t/t^*)} s^{t/t^*-1} e^{-\alpha s}, \quad s > 0, \alpha > 0, t^* > 0.$$

- ▶ We set the parameter $\alpha = \frac{1}{t^*}$, so that the gamma process becomes unbiased, i.e. $E^{\mathbb{Q}}\Gamma_t = t$ for all $t \geq 0$.
- ▶ The PDF of the unbiased gamma process Γ at time $t \geq 0$ is:

$$\mathbb{Q}\{\Gamma_t \in ds\} = \frac{s^{t/t^*-1} e^{-s/t^*}}{(t^*)^{t/t^*} \Gamma(t/t^*)}, \quad s > 0,$$

for parameter $t^* > 0$.

- ▶ When $t = t^*$, this PDF is exponential. We choose t^* to be the given term, so that the distribution of the gamma clock at expiry is exponential.

- ▶ We assume that the underlying spot price process S is obtained by subordinating the driftless diffusion D to the unbiased gamma process Γ :

$$S_t = D_{\Gamma_t}, \quad t \geq 0.$$

- ▶ S inherits the local martingale property from the diffusion D .
- ▶ S inherits dependence of its increments on its level from D .
- ▶ S inherits jumps from the gamma process Γ .
- ▶ S inherits time homogeneity from D and Γ .

Forward PDDE

- ▶ Define $C(S, T, K)$ as the value of a European call with stock price $S \in \mathbb{R}$, term $T \geq 0$, and strike $K \in \mathbb{R}$ in the LVG model.
- ▶ Then it can be shown that the LVG model value satisfies the following forward PDDE:

$$\frac{C(S, T + t^*, K) - C(S, T, K)}{t^*} = \frac{a^2(K)}{2} \frac{\partial^2}{\partial K^2} C(S, T + t^*, K),$$

for all $S \in \mathbb{R}$, $T \geq 0$, and $K \in \mathbb{R}$.

- ▶ Remarkably, the effect of the gamma time change is to simultaneously inject realism into the underlying price dynamics and to semi-discretize Dupire's forward PDE.
- ▶ More specifically, Dupire's infinitesimal calendar spread gets replaced by a discrete calendar spread, with the time between the two maturities given by t^* .

Forward PDDE (Con'd)

- ▶ Setting the first maturity date to the valuation date and $S = S_0$, the LVG model value of a call satisfies:

$$\frac{C(S_0, t^*, K) - (S_0 - K)^+}{t^*} = \frac{a^2(K)}{2} \frac{\partial^2}{\partial K^2} C(S_0, t^*, K),$$

for all $K \in \mathbb{R}$.

- ▶ If call prices at a continuum of strikes are given, then the diffusion coefficient $a(x)$ becomes observed.
- ▶ As a result, the stock price process becomes specified. Options of other terms and other derivatives can be valued by Monte Carlo.

- ▶ Consider a European-style claim paying $\phi(S_T)$ at its maturity date $T \geq t^*$.
- ▶ Let $V(S, t) : \mathbb{R} \times [0, T] \mapsto \mathbb{R}$ denote the function that relates the value of this claim to the underlying spot price S and calendar time t .
- ▶ It can be shown that $V(S, t)$ solves the following backward PDDE:

$$\frac{\sigma^2(S)}{2} \frac{\partial^2}{\partial S^2} V(S, t) = \frac{V(S, t) - V(S, t + t^*)}{t^*}.$$

- ▶ We note that this PDDE arises if Rothe's method (also known as semi-discretization or (horizontal) method of lines) is used to discretize the diffusion equation.

Backward PDDE (Con'd)

- ▶ The European claim value function V satisfies the terminal condition:

$$V(S, T) = \phi(S), \quad S \in \mathbb{R}.$$

- ▶ Re-arranging the backward PDDE implies that $V(S, t)$ solves the following second order inhomogeneous ODE:

$$t^* \frac{a^2(S)}{2} \frac{\partial^2}{\partial S^2} V(S, t) - V(S, t) = -V(S, t + t^*).$$

- ▶ Evaluating at $t = T - t^*$ the RHS forcing term is $\phi(S)$. One can numerically solve the ODE and recursively step backwards in time steps of length t^* .
- ▶ The algorithm switches from the express (ODE) to the local (PIDE), when the express would overshoot the valuation time.

- ▶ Setting $T = 0$ and $S = S_0$ in the forward PDDE, the LVG model value of a call solves a linear second order ODE in K :

$$t^* \frac{a^2(K)}{2} \frac{\partial^2}{\partial K^2} C(S_0, t^*, K) - C(S_0, t^*, K) = -(S_0 - K)^+$$

for all $K \in \mathbb{R}$.

- ▶ If call prices at only a finite number of strikes are given, then we can no longer determine the a function explicitly.
- ▶ As the next best thing, we assume that the diffusion coefficient a is piecewise constant. As a result, between any 2 adjacent strikes, the linear 2nd order ODE has constant coefficients and given Dirichlet boundary conditions.
- ▶ For given a , we have an analytic solution using sinh and cosh.

Discrete Strikes (Con'd)

- ▶ As the constant diffusion coefficient a is not known ex ante, we assume that for the 3 strikes nearest the forward, K_{-1} , K_0 , and K_1 , a is constant between K_{-1} and K_1 .
- ▶ Using univariate root finding, we can solve for this constant a , and get call prices for all strikes $K \in [K_{-1}, K_1]$.
- ▶ Looking at the next highest strike K_2 , we can again solve a two point boundary value problem analytically with a given. As a is not given, we root find so as to match slopes of C in K .
- ▶ Proceeding sequentially, we can determine all of the a levels and produce C^1 call prices for all K .

Further Properties of the LVG Model

- ▶ Although the diffusion coefficient $a(x)$ is only piecewise continuous, the call prices produced by the LVG model are piecewise C^2 in strike K . Likewise, derivative security prices produced by the model are piecewise C^2 in spot S . Option deltas, thetas, and (left and right) gammas always have the conventional sign.
- ▶ At listed maturities and corresponding calendar times, many LVG contingent claim values can be analytically expressed. Between such times, “exact” valuation requires numerical solution of a PIDE.

Comparison to Similar Approaches & Calibration of LVG

- ▶ The LVG model, Entropy Methods, and Mixtures of Lognormals also produce continuous-time arbitrage-free dynamics for the underlying asset price that are exactly consistent with any given set of option prices at discrete strikes and terms.
- ▶ In terms of smoothness in K, T, S, t , LVG lies between Entropy Methods and Mixtures of Lognormals.
- ▶ Calibration of the LVG process just involves univariate root-finding one option at a time. In contrast, Entropy Methods and Mixtures of Lognormals both require global searches, which can take a long time or may never succeed.

Parting Remarks

- ▶ Just as the gamma function is a natural way to interpolate (and extrapolate) the factorials, subordination to a gamma process is a natural way to interpolate option prices at discrete terms.
- ▶ There is no doubt in my mind that LVG can be improved upon in several dimensions, eg. realism, at the possible cost of sacrificing in other dimensions, eg. simplicity.
- ▶ In my view, the automated options market making problem remains open.