Backward SDEs of mean-field type and a related control problem

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The mean-field approximation may be useful in problems such as:

- Price formation and dynamic equilibria (Aumann (1964),..., Föllmer (95'),..., Larsy and Lions (00'-07'), G. Carmona (07'));

- Formation of volatility (due to trading impacts), auctions and insider trading, stock pinning (Kyle (85'), Avellaneda & Lipkin (03')).

The list is much longer..

The talk is based on the following papers

- R. Buckdahn, B. Djehiche, J. Li and S. Peng (2009): Mean-Field Backward Stochastic Differential Equations. A Limit Approach. To appear in the Annals of Probability.

- D. Andersson and B. Djehiche (2009): A maximum principle for SDEs of mean-field type (Preprint).

Forward SDEs of mean-field type

The dynamics of an SDE of mean-field type on $\mathbb{R}$ is

$$\begin{cases} dx_t &= b\left(t, x_t, \mathbb{E}\psi\left(x_t\right)\right) dt + \sigma\left(t, x_t, \mathbb{E}\phi\left(x_t\right)\right) dB_t. \\ x_0 &= x^0, \end{cases} \tag{1}$$

The classical example is the McKean-Vlasov model, in which the coefficients are linear in the marginal law $(P_{x_t}(dy))$ of the process (see e.g. Sznitman (1989) and the references therein):

$$\begin{cases} dx_t &= \int_{\mathbb{R}} b\left(t, x_t, y\right) P_{x_t}(dy) dt + \int_{\mathbb{R}} \sigma\left(t, x_t, y\right) P_{x_t}(dy) dB_t. \\ x_0 &= x^0, \end{cases} \tag{2}$$

For the nonlinear case, see Jourdain *et al. (2008).*

A related optimal control problem

The dynamics of the controlled SDE of mean-field type on $\mathbb{R}$ is

$$\begin{cases} dx_t & = b\left(t, x_t, \mathbb{E}\psi\left(x_t\right), u_t\right) dt + \sigma\left(t, x_t, \mathbb{E}\phi\left(x_t\right), u_t\right) dB_t. \\ x(0) & = x_0, \end{cases} \tag{3}$$

The cost functional is of the form

$$J(u) = \mathbb{E}\left(\int_0^T h\left(t, x_t, \mathbb{E}\varphi\left(x_t\right), u_t\right) dt + g\left(x_T, \mathbb{E}\chi\left(x_T\right)\right)\right). \tag{4}$$

It is also of mean-field type, as the functions h and g depend on the law of the state process.

We want to "find" or characterize (through a maximum principle)

$$u^* = \arg\min_u J(u).$$

Time inconsistent control problem

The fact that g is **nonlinear in** $\mathbb{E}_{\mathcal{X}}(x_t)$ makes the problem time inconsistent.

The classical **Bellman optimality principle** based on the law of iterated expectations on J **does not hold**.

An example: Mean-variance portfolio selection

The dynamics of the self-financing portfolio is

$$dx_t = (\rho_t x_t + (\alpha_t - \rho_t)\, u_t)\, dt + \sigma_t u_t dB_t, \;\; x_0 = x(0). \tag{5}$$

The control u_t denotes the amount of money invested in the risky asset at time t.

The cost functional, to be minimized, is given by

$$J(u) = \frac{\gamma}{2}\mathrm{Var}(x_T) - \mathbb{E}x_T. \tag{6}$$

By rewriting it as

$$J(u) = \mathbb{E}\left(\frac{\gamma}{2}x_T^2 - x_T\right) - \frac{\gamma}{2}\left(\mathbb{E}x_T\right)^2,$$

it becomes of type (2) i.e. nonlinear in $\mathbb{E}x_T$.

Previous work: extend the HJB equation to the mean-field case

(1) Ahmed and Ding (2001) express the value function in terms of the Nisio semigroup of operators and derive a (very complicated) HJB equation.

(2) Huang *et al.* (2006) use the Nash Certainty Equivalence Principle to solve an extended HJB equation.

(3) Lasry and Lions (06'-07') suggest a new class of nonlinear HJB involving the dynamics of the probability laws $(\mu_t)_t$.

(4) Björk and Murgoci (2008) use the notion of Nash equilibrium to transform the time inconsistent control problem into a standard one and derive an "extended" HJB equation.

The mean-field SDE is obtained as an L^2-limit

$$\mathbb{E}\left[\sup_{t\in[0,T]} |x_t^{i,n} - x_t|^2\right] \to 0, \text{ as } n \to \infty$$

of a system of **interacting** particles of the form

$$\begin{cases} dx_t^{i,n} & = b\left(t, x_t^{i,n}, \frac{1}{n}\sum_{j=1}^n \psi\left(x_t^{j,n}\right)\right)dt + \sigma\left(t, x_t^{i,n}, \frac{1}{n}\sum_{j=1}^n \phi\left(x_t^{j,n}\right)\right)dB_t^i, \\ x_0^{i,n} & = x^0, \end{cases} \tag{7}$$

B^i, $i = 1, 2, \ldots, n$ being independent Brownian motions.

Notice: $x^{i,n}, i = 1, \ldots, n$ are **identically distributed** for all $n \geq 1$.

For the general case, where the coefficients depend on the law of x_t, one may use:

$$\begin{cases} dx_t^{i,n} & = b\left(t, x_t^{i,n}, \frac{1}{n}\sum_{j=1}^n \delta_{x_t^{j,n}}\right)dt + \sigma\left(t, x_t^{i,n}, \frac{1}{n}\sum_{j=1}^n \delta_{x_t^{j,n}}\right)dB_t^i, \\ x_0^{i,n} & = x^0, \end{cases}$$

Standard backward SDEs

$(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space, on which B is a standard Brownian motion. $\mathbf{F} := (\mathcal{F}_t)_t = \mathbf{F}^B \vee \mathcal{N}_{\mathbb{P}}$ (i.e. augmented by all $\mathbb{P}$-null sets).

$$\mathcal{S}^2_{\mathbf{F}} = \left\{(Y_t)_t,\ \mathbb{R}-\text{valued, contin. adapted}: \ \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^2\right] < \infty\right\},$$

$$\mathbf{L}^2_{\mathbf{F}} = \left\{(Z_t)_t,\ \mathbb{R}-\text{valued, prog. measur.}: \ \mathbb{E}\left[\int_0^T |Z_t|^2 dt\right] < \infty\right\},$$

Let $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be $\mathbf{F}$-measurable and satisfy

(A1) $\quad |g(t, y_1, z_1) - g(t, y_2, z_2)| \le C(|y_1 - y_2| + |z_1 - z_2|),$

(A2) $\quad g(\cdot, 0, 0) \in \mathbf{L}^2_{\mathbf{F}}.$

Lemma (Pardoux and Peng (1990)) Let g satisfy (A1) and (A2). Then for any $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, the backward SDE

$$y_t = \xi + \int_t^T g(s, \omega, y_s, z_s)ds - \int_t^T z_s dB_s, \quad 0 \le t \le T,$$

has a unique solution $(y, z) \in \mathcal{S}^2_{\mathbf{F}} \times \mathbf{L}^2_{\mathbf{F}}$.

Connection with PDEs

Consider a forward SDE

$$\begin{cases} x_s^{t,x} = x + \int_t^s b\left(r, x_r^{t,x}\right) dr + \int_t^s \sigma\left(r, x_r^{t,x}\right) dB_r, \ \forall x, \ \ s \geq t, \\ \\ x_s^{t,x} = x, \quad \ s \leq t, \end{cases} \tag{8}$$

with infinitesimal generator:

$$\mathcal{L}\varphi(x) = \frac{1}{2}\sigma^2(t,x)\varphi''(x) + b(t,x)\varphi'(x).$$

When $\xi = \Phi(x_T^{t,x})$, given the solution of the following BSDE

$$y_s = \Phi(x_T^{t,x}) + \int_s^T g(r, x_r^{t,x}, y_r, z_r)dr - \int_s^T z_r dB_r, \ \ 0 \leq s \leq T,$$

There exists a continuous deterministic function $v(t,x)$ such that

$$Y_s = v(s, x_s^{t,x}); \quad Z_s = "\,\sigma(s, x_s^{t,x}) \partial_x v(s, x_s^{t,x})", \quad s \geq t,$$

and which is a viscosity solution of the semi-linear PDE

$$\partial_t v(t,x) + \mathcal{L} v(t,x) + g(t, x, v(t,x), \sigma(t,x) \partial_x v(t,x)) = 0, \quad v(T,x) = \Phi(x).$$

Backward SDEs of mean-field type

Recall the forward SDE of McKean-Vlasov type

$$\begin{cases} dX_t &= \int_{\mathbb{R}} b\,(t, X_t, x)\, P_{X_t}(dy)dt + \int_{\mathbb{R}} \sigma\,(t, X_t, y)\, P_{X_t}(dx)dB_t. \\ \\ X_0 &= x^0, \end{cases} \tag{9}$$

whose infinitesimal generator is

$$\mathcal{L}(m)\varphi(x) = \frac{1}{2}\left(\int \sigma(t,x,y)m(dy)\right)^2 \varphi''(x) + \left(\int b(t,x,y)m(dy)\right)\varphi'(x),$$

$m(dy)$ being a probability measure.

The McKean-Vlasov (or Fokker-Planck) equation reads:

$$\begin{cases} \partial_t m_t - \mathcal{L}_t^*(m_t)m_t = 0, \\ \\ m_0(dy) = \text{probability measure.} \end{cases} \tag{10}$$

We will consider the following BSDE of McKean-Vlasov type, driven by the process X, whose unique solution $\Lambda = (X, Y, Z)$ satisfies

$$Y_t = \int_{\mathbb{R}} \Phi(X_T, x) P_{X_T}(dx) + \int_0^T \int_{\mathbb{R}^{1+1+1}} f(s, \Lambda_s, \lambda) P_{\Lambda_s}(d\lambda)\, ds - \int_t^T Z_s dB_s,$$

and should be adapted to **the filtration of the Brownian motion $\mathbb{F}$**, under certain conditions on the coefficients b, σ and

$$f : \mathbb{R}^{1+1+1} \times \mathbb{R}^{1+1+1} \to \mathbb{R}, \quad \Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}.$$

Connection with a class of PDEs suggested by Lasry & Lions (06)

Meta-theorem: Given the BSDE

$$Y_t = \int_{\mathbb{R}} \Phi(X_T, x) P_{X_T}(dx) + \int_0^T \int_{\mathbb{R}} f(s, \Lambda_s, x) P_{X_s}(dx)\, ds - \int_t^T Z_s dB_s,$$

There exists a continuous deterministic function $v(t,x)$ such that

$$Y_s = v(s, X_s^{t,x}), \;\; Z_s = \text{"} \left(\int \sigma(s, X_s^{t,x}, y) m_s(dy) \right) \partial_x v(s, X_s^{t,x}) \text{"},$$

such that v (depending on m) is a viscosity solution to the following PDE

$$\begin{cases} \partial_t v(t,x) + \mathcal{L} v(t,x) + \left(\int f(t,x,v(t,x), (\sigma(s,x,z) m_s(dz))\, \partial_x v(t,x) \right) m_s(dy) = 0, \\ v(T,x) = \int \Phi(T,x,y) m_T(dy), \\ \partial_t m_t - \mathcal{L}_t^*(m_t) m_t = 0, \\ m_0(dy) = \text{probability measure}. \end{cases}$$

To solves this equation we need an appropriate approximation scheme. Using an approximation scheme similar to Eq. (7), for the forward case, we may lose adaptation of the solutions with respect to $\mathbf{F}$, due to the interaction between the particles that are not necessarily driven by the same underlying Brownian motion B.

We need to construct a sequence of processes $\Lambda^n = (X^n, Y^n, Z^n)$ that solve ordinary forward and backward SDEs, are adapted to the **the filtration of the Brownian motion** B, and which converges properly to $\Lambda = (X, Y, Z)$.

Main result

Assume

(B1) b, σ bounded and Lipschitz

(B2) f bounded, $|f(t,(u_1,v_1)) - f(t,(u_2,v_2))| \leq C(|u_1 - u_2| + |v_1 - v_2|)$,

(B3) Φ bounded, $|\Phi(x_1,\widehat{x}_1) - \Phi(x_2,\widehat{x}_2)| \leq C(|x_1 - x_2| + |\widehat{x}_1 - \widehat{x}_2|)$,

then

• there exists a unique solution $(X,Y,Z) \in \mathcal{S}^2_{\mathbf{F}} \times \mathcal{S}^2_{\mathbf{F}} \times \mathbf{L}^2_{\mathbf{F}}$,

• there exists a sequence of processes $\Lambda^n = (X^n, Y^n, Z^n)$ that solve ordinary forward and backward SDEs, are adapted to the **the filtration of the Brownian motion** B such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X^n_t - X_t|^2 + \sup_{t\in[0,T]} |Y^n_t - Y_t|^2 + \int_0^T |Z^n_t - Z_t|^2 dt\right] \leq C/n.$$

The sequence $\Lambda^n = (X^n, Y^n, Z^n)$ reads:

$$\begin{cases} X_t^n &= x^0 + \int_0^t \frac{1}{n} \sum_{k=1}^n b(s, X_s^n, \Theta^k(X_s^n))ds + \int_0^t \frac{1}{n} \sum_{k=1}^n \sigma(s, X_s^n, \Theta^k(X_s^n))dB_s; \\ Y_t^n &= \frac{1}{n} \sum_{k=1}^n \Phi(X_T^n, \Theta^k(X_T^n)) + \int_t^T \frac{1}{n} \sum_{k=1}^n f(s, \Lambda_s^n, \Theta^k(\Lambda_s^n))ds - \int_t^T Z_s^n dB_s, \end{cases}$$

where, the family of shift operators $\Theta^k : \tilde{\Omega} \to \tilde{\Omega};\ k \geq 0$ is constructed such that for all $\xi \in L^0(\Omega, \mathcal{F}, \mathbb{P})$:

• the r.v. $\Theta^k(\xi)$, $k \geq 1$, are **independent and identically distributed**, of the same law as ξ and **independent** of the driving Brownian motion B (very important for simulations). Note that $\Theta^0(\xi) = \xi$.

or, equivalently,

• the shift operators Θ^k, $k \geq 1$, leave the Wiener measure P on $\tilde{\Omega}$ invariante:

$$P \otimes [\Theta^k]^{-1} = P,$$

where, the original probability space is one component in the infinite product of (independent) Wiener spaces:

$$\tilde{\Omega} = (C_0([0,T];\mathbb{R}))^I, \quad P = \text{products of } \mathbb{P},$$

where, I is a countable set of indexes....

See (Buckdahn *el al.* (2009)) for details.

The optimal control problem

The dynamics of the controlled SDE of mean-field type on $\mathbb{R}$ is

$$\begin{cases} dx_t & = b\left(t, x_t, \mathbb{E}\psi\left(x_t\right), u_t\right) dt + \sigma\left(t, x_t, \mathbb{E}\phi\left(x_t\right), u_t\right) dB_t. \\ x(0) & = x_0, \end{cases} \tag{11}$$

The cost functional is of the form

$$J(u) = \mathbb{E}\left(\int_0^T h\left(t, x_t, \mathbb{E}\varphi\left(x_t\right), u_t\right) dt + g\left(x_T, \mathbb{E}\chi\left(x_T\right)\right)\right). \tag{12}$$

It is also of mean-field type, as the functions h and g depend on the law of the state process.

We want to "find" or characterize (through a maximum principle)

$$u^* = \arg\min_u J(u).$$

A maximum principle for controlling mean-field type SDEs

The dynamics of the controlled SDE of mean-field type on $\mathbb{R}$ is

$$\begin{cases} dx_t & = b\left(t, x_t, \mathbb{E}\psi\left(x_t\right), u_t\right) dt + \sigma\left(t, x_t, \mathbb{E}\phi\left(x_t\right), u_t\right) dB_t, \\ x(0) & = x_0, \end{cases}$$

The cost functional is of the form

$$J(u) = \mathbb{E}\left(\int_0^T h\left(t, x_t, \mathbb{E}\varphi\left(x_t\right), u_t\right) dt + g\left(x_T, \mathbb{E}\chi\left(x_T\right)\right)\right).$$

We want to "find/characterize"

$$u^* = \arg\min_{u \in \mathcal{U}} J(u),$$

$\mathcal{U}$ is the class of measurable, adapted processes $u : [0, T] \times \Omega \longrightarrow U$.

Assumptions

(1) The action space U is a closed and **convex** subset of $\mathbb{R}$.

(2) All the involved functions are sufficiently smooth.

Let $\widehat{u}$ denote an optimal control, and $\widehat{x}$ the corresponding state process. Also, denote

$$\widehat{b}(t) = b\left(t, \widehat{x}_t, \mathbb{E}\widehat{\psi}(t), \widehat{u}_t\right),$$
$$\widehat{\psi}(t) = \psi\left(\widehat{x}_t\right),$$

and similarly for the other functions and their derivatives.

Necessary conditions for optimality a la Bensoussan (1982)

(1) **Taylor expansions**.

let x_t^θ correspond to $u_t^\theta = \widehat{u}_t + \theta v_t,\ v_t \in \mathcal{U}$.

Lemma. Let

$$\begin{cases} dz_t = & \left(\widehat{b}_x(t) z_t + \widehat{b}_y(t)\mathbb{E}\left(\widehat{\psi}_x(t) z_t\right) + \widehat{b}_v(t) v_t\right) dt \\ & + \left(\widehat{\sigma}_x z_t + \widehat{\sigma}_y(t)\mathbb{E}\left(\widehat{\phi}_x(t) z_t\right) + \widehat{\sigma}_v(t) v_t\right) dB_t. \\ z_0 = & 0. \end{cases}$$

Then it holds that

$$\lim_{\theta\to 0} \mathbb{E} \sup_{0\le t\le T} \left| \frac{x_t^\theta - \widehat{x}_t}{\theta} - z_t \right|^2 = 0.$$

Lemma. The Gateaux derivative of the cost functional J is given by

$$\frac{d}{d\theta} J\left(\widehat{u}+\theta v\right)\Big|_{\theta=0} = \mathbb{E}\left(\int_0^T \left(\widehat{h}_x(t) z_t + \widehat{h}_y(t)\mathbb{E}\left(\widehat{\varphi}_x(t) z_t\right) + \widehat{h}_v(t) v_t\right) dt\right)$$

$$+\mathbb{E}\left(\widehat{g}_x(T) z_T + \widehat{g}_y(T)\mathbb{E}\left(\chi_x(T) z_T\right)\right).$$

(2) **Duality**. The adjoint equation

$$\left\{\begin{aligned} d\widehat{p}_t = & -\left(\widehat{b}_x(t)\widehat{p}_t + \widehat{\sigma}_x(t)\widehat{q}_t + \widehat{h}_x(t)\right)dt + \widehat{q}_t dB_t \\ & -\left(\mathbb{E}\left(\widehat{b}_y(t)\widehat{p}_t\right)\widehat{\psi}_x(t) + \mathbb{E}\left(\widehat{\sigma}_y\widehat{q}_t\right)\widehat{\phi}_x(t) + \mathbb{E}\left(\widehat{h}_y(t)\right)\widehat{\varphi}_x(t)\right)dt, \\ \widehat{p}_T = & \ \widehat{g}_x(T) + \mathbb{E}\left(\widehat{g}_y(T)\right)\widehat{\chi}_x(T). \end{aligned}\right. \tag{13}$$

Under our assumptions, this is a linear mean-field backward SDE with bounded coefficients. It has has a unique adapted solution such that

$$\mathbb{E}[\sup_{t\in[0,T]} |\widehat{p}|_T^2] + \mathbb{E}\int_0^T |\widehat{q}_t|^2\, dt < +\infty. \tag{14}$$

(See Buckdahn *et al. (2007), Theorem 3.1)*

Lemma *(relation between $\widehat{p}_t$ and z_t)*

$$\mathbb{E}\left(\widehat{p}_T z_T\right) = \mathbb{E}\left(\int_0^T \left(\widehat{p}_t\widehat{b}_v(t)v_t - z_t\widehat{h}_x(t) - z_t\mathbb{E}\left(\widehat{h}_y(t)\right)\widehat{\varphi}_x(t) + \widehat{q}_t\widehat{\sigma}_v(t)v_t\right)dt\right).$$

Define the usual Hamiltonian

$$
\begin{aligned}
H\left(x_t, u_t, p_t, q_t\right) := & h\left(t, x_t, \mathbb{E}\left(\varphi(x_t)\right), u_t\right) + b\left(t, x_t, \mathbb{E}\left(\psi(x_t)\right), u_t\right) p_t \\
& + \sigma\left(t, x_t, \mathbb{E}\left(\phi(x_t)\right), u_t\right) q_t.
\end{aligned}
$$

Noting that

$$
\mathbb{E}\left(\widehat{p}_T z_T\right) = \mathbb{E}\left(\widehat{g}_x(T) z_T + \widehat{g}_y\left(T\right) \mathbb{E}\left(\widehat{\chi}_x(T) z_T\right)\right),
$$

we obtain the following result.

Proposition The Gateaux derivative of the cost functional can be expressed in terms of the Hamiltonian H in the following way.

$$
\begin{aligned}
\frac{d}{d\theta} J\left(\widehat{u} + \theta v\right)\Big|_{\theta=0} = & \mathbb{E}\left(\int_0^T \left(\widehat{h}_v(t) v_t + \widehat{p}_t \widehat{b}_v(t) v_t + \widehat{q}_t \widehat{\sigma}_v(t) v_t\right) dt\right) \\
= & \mathbb{E}\left(\int_0^T \frac{d}{dv} H\left(\widehat{x}_t, \widehat{u}_t, \widehat{p}_t, \widehat{q}_t\right) v_t dt\right).
\end{aligned}
$$

Theorem Under our 'assumptions', if $\widehat{u}_t$ is an optimal control with state trajectory $\widehat{x}_t$, then there exists a pair $(\widehat{p}_t, \widehat{q}_t)$ of adapted processes which satisfies (13) and (14), such that

$$\frac{\mathrm{d}}{\mathrm{d}v} H\left(t, \widehat{x}_t, \widehat{u}_t, \widehat{p}_t, \widehat{q}_t\right)\left(v - \widehat{u}_t\right) \geq 0, \ \mathbb{P}-\text{a.s., for all } t \in [0, T]. \tag{15}$$

This condition is also sufficient, under further assumption on the coefficients.

A worked out example- Mean-variance portfolio selection

The state process equation is

$$dx_t = (\rho_t x_t + (\alpha_t - \rho_t)\, u_t)\, dt + \sigma_t u_t dB_t, \;\; x_0 = x(0). \tag{16}$$

The cost functional, to be minimized, is given by

$$J(u) = \mathbb{E}\left(\frac{\gamma}{2} x_T^2 - x_T\right) - \frac{\gamma}{2}\left(\mathbb{E} x_T\right)^2,$$

The Hamiltonian for this system is

$$H(t, x, u, p, q) = (\rho_t x + (\alpha_t - \rho_t)\, u)\, p + \sigma_t u q.$$

The adjoint equation becomes

$$\begin{cases} dp_t &= -\rho_t p_t dt + q_t dB_t, \\ p_T &= \gamma(x_T - \mu_T) - 1, \end{cases}$$

Try a solution of the form (with A_t, C_t deterministic functions)

$$\begin{cases} p_t & = A_t\left(x_t - \mu_t\right) - C_t, \\ \\ A_T & = \gamma,\ C_T = 1 \end{cases}$$

After easy manipulations we get

$$\begin{cases} A_t & \left(\rho_t\left(x_t - \mu_t\right) + \left(\alpha_t - \rho_t\right)\left(u_t - \mathbb{E}\left(u_t\right)\right)\right) + A_t'\left(x_t - \mu_t\right) - C_t' \\ \\ & = -\rho_t A_t\left(x_t - \mu_t\right) + \rho_t C_t, \\ \\ q_t & = A_t \sigma_t u_t. \end{cases}$$

together with the first order condition for minimizing the Hamiltonian yielding

$$\left(\alpha_t - \rho_t\right) p_t + \sigma_t q_t = 0.$$

A solution of the mean-variance portfolio selection problem is given by

$$\widehat{u}\,(t,\widehat{x}_t) = \frac{\alpha_t - \rho_t}{\sigma_t^2}\left(x_0 e^{\int_0^t \rho_s ds} + \frac{1}{\gamma} e^{\int_0^T \Lambda_s ds - \int_t^T \rho_s ds} - \widehat{x}_t\right),$$

where,

$$\Lambda_t = \frac{(\rho_t - \alpha_t)^2}{\sigma_t^2}.$$

which is identical to the optimal control found in Zhou and Li (2000) (Eqs. (5.12), (6.7) and the subsequent comments) by embedding it into a stochastic LQ problem.