ON NONLINEAR MODELS OF MARKETS WITH FEEDBACK DUE TO FINITE LIQUIDITY: SOME CAUTIONARY NOTES

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PDEs and Mathematical Finance III

OUTLINE OF TALK

Motivation

Black-Scholes model assumes an infinitely elastic market in the underlying, i.e. trading the underlying does not impact on its price. Have been many attempts to relax this (unrealistic) assumption - many result in highly nonlinear PDEs. One (canonical) version (Frey, 1998, Sircar & Papanicolaou, 1998, Schönbucher & Wilmott, 2000) will be assessed, mostly from the PDE perspective, but with some SDE results thrown in. Used extensively in the past, *with fudges*

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THE MODEL - HEURISTIC DERIVATION (C.F. LIPTON, 2001)

Given any stochastic process we add another term to the SDE to account for the effect of price impact, i.e.

$$dS = \mu(S,t)dt + \sigma(S,t)dW_t + \lambda(S,t)df$$

where $\lambda(S,t)$ is arbitrary, and W_t is a Wiener process, and $f(S,t)$ models price impact. Applying Itô's formula to $f(S,t)$

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2 + \ldots$$

Substitution gives (as $dt \to 0$)

$$\left(1 - \lambda\frac{\partial f}{\partial S}\right) dS = \left(\mu + \lambda\frac{\partial f}{\partial t}\right) dt + \frac{\lambda}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2 + \sigma dW_t + \ldots,$$

and so squaring yields $(dS)^2 = \frac{\sigma^2 dt}{\left(1-\lambda\frac{\partial f}{\partial S}\right)^2} + o(dt)$.

HEURISTIC DERIVATION (CONTINUED)

Substitution and a little rearranging gives

$$dS = \hat{\mu}(S,t)dt + \hat{\sigma}(S,t)dW_t$$

where

$$\begin{aligned}
\hat{\mu}(S,t) &= \frac{\hat{\sigma}}{\sigma}\left[\mu + \lambda\left(\frac{\partial f}{\partial t} + \frac{1}{2}\hat{\sigma}^2\frac{\partial^2 f}{\partial S^2}\right)\right] \\
\hat{\sigma}(S,t) &= \frac{\sigma}{\left(1 - \lambda\frac{\partial f}{\partial S}\right)}
\end{aligned}$$

Hence price impact function modifies original stochastic process (in particular volatility)

HEURISTIC DERIVATION (CONTINUED)

Leads to the **Generalised Black-Scholes Equation**

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\frac{\sigma^2 S^2}{\left(1 - \lambda \frac{\partial f}{\partial S}\right)^2}\frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV = 0$$

where $\tau = T - t$ and geometric Brownian motion has been used for the original stochastic process.
But what are the functions $\lambda(S, \tau)$ and $f(S, \tau)$?
Can regard $\lambda(S, \tau)$ as a function dependent on how we choose to model liquidity on a market micro-structure level.

$f(S, \tau)$ is identified as the number of shares traded due to some (deterministic) trading strategy; for example a buy and hold gives $f(S, \tau) = \text{const.}$.

HEURISTIC DERIVATION (CONTINUED)

If we are interested in the price impact due to delta hedging then can identify f as the delta of the option being replicated, i.e.

$$f = \Delta = \frac{\partial V^*}{\partial S}.$$

This leads to the question as to what strategy the hedgers are assumed to follow - consider two courses of action:

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This leads to the question as to what strategy the hedgers are assumed to follow - consider two courses of action:

1. They use the *Black-Scholes* option delta to hedge, ignoring price impact.
2. Or they try to incorporate price impact into the hedging strategy by using the modified delta.

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The first case we shall call first-order feedback and leads to the **linear** equation

$$V_\tau - \frac{\sigma^2 S^2 V_{SS}}{2\left(1 - \lambda V_{SS}^{BS}\right)^2} - rSV_S + rV = 0,$$

where V^{BS} is the solution to the Black-Scholes equation ($\lambda = 0$).

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The second case we shall call full feedback and leads to the interesting **nonlinear** equation

$$V_\tau - \frac{\sigma^2 S^2 V_{SS}}{2\left(1-\lambda V_{SS}\right)^2} - rSV_S + rV = 0.$$

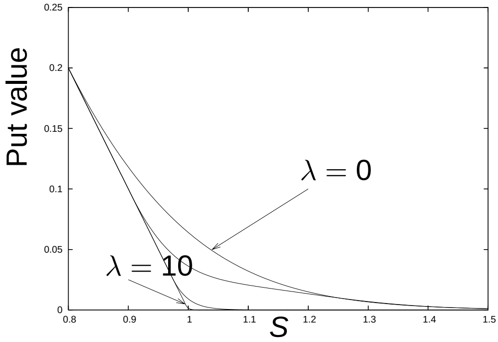
We investigate the properties of both of these equations.

EARLY EXERCISE PUT PROBLEM (FIRST-ORDER FEEDBACK)

Using standard PSOR method

EARLY EXERCISE PUT PROBLEM (FIRST-ORDER FEEDBACK)

Using standard PSOR method no problem!



Value of American put options, $T = 1$, $r = 0.04$, $\sigma = 0.2$, $X = 1$ and for $\lambda = 0, 1, 2, 5, 10$; the variation with λ is monotonic

LOCAL ANALYSIS CLOSE TO EXPIRY - FIRST-ORDER FEEDBACK

Although ‘full’ results suggest little difference from the standard ($\lambda = 0$) case, there are some subtle differences:
Period close to expiry generally the most intricate in the lifetime of an option: seek a solution of the form (for European types)

$$V = \tau g(\xi), \quad \text{where} \quad \xi = \frac{S - X}{\tau}.$$

Note all ‘the action’ occurs in a region $O(\tau)$, in asset space S, close to the strike and is thinner than for classical Black-Scholes which occurs in a region $O(\sqrt{\tau})$.
For a put:

$$g(\xi) = (\xi + rX)\left[\Phi\left(\frac{\xi + rX}{\kappa}\right) - 1\right] + \frac{\kappa}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\xi + rX}{\kappa}\right)^2}$$

where $\kappa = \frac{\sqrt{\pi}\sigma^2 X^2}{\lambda}$, $\Phi(\cdot)$ is standard normal cumulative distribution function

EARLY-EXERCISE PUTS

For the classical Black-Scholes problem the η-scaling region $O(\sqrt{\tau})$ does not capture the free boundary of early-exercise options, (it occurs on a longer scale, $O(\sqrt{-\tau\log(\tau)})$). However for first-order feedback equation the ξ-scaling region $O(\tau)$ DOES encompass the free boundary.

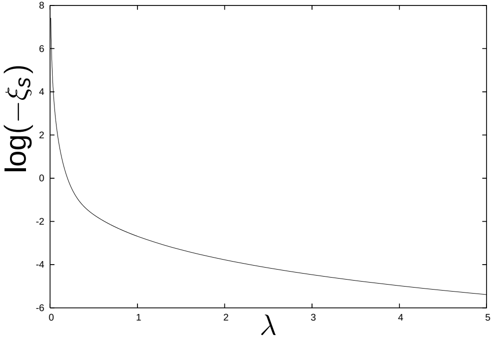
Modified conditions (ξ_s is the free boundary)

$$g \to 0 \text{ as } \xi \to \infty \quad g = -(\xi + rX) \text{ and } g_\xi = -1 \text{ on } \xi = \xi_s$$

Location of the free boundary ξ_s given by solution to

$$\Phi\left(\frac{\lambda(\xi_s + rX)}{\sqrt{\pi}\sigma^2 X^2}\right) = 1 - \frac{\sigma^2 X}{\sqrt{2r}} \exp\left(-\frac{\lambda^2}{2\pi}\left(\frac{\xi_s + rX}{\sigma^2 X^2}\right)^2\right)$$

LOCATION OF FREE BOUNDARY CLOSE TO EXPIRY - FIRST-ORDER FEEDBACK

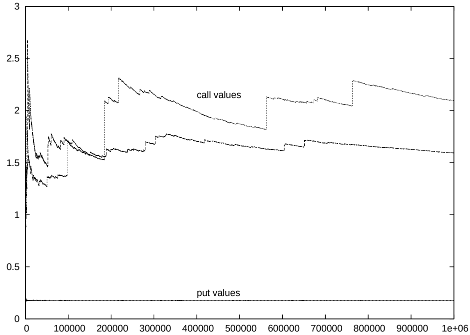


$X = 1$, $r = 0.04$, $\sigma = 0.2$

Asymptotic analysis shows $\xi_s \to -\infty$ as $\lambda \to 0$ - consistent with the fact that free boundary is on a relatively longer scale for regular Americans

FIRST-ORDER FEEDBACK USING THE SDE (SIMULATION) APPROACH

- Set $\hat{\mu} = r$, V_{SS}^{BS} from analytic form.
- $S = 1$, $\sigma = 0.2$, $T = 10$, $X = 1$, $r = 0.04$, $\lambda = 1$ (European style)



Two batches of 10^6 simulations

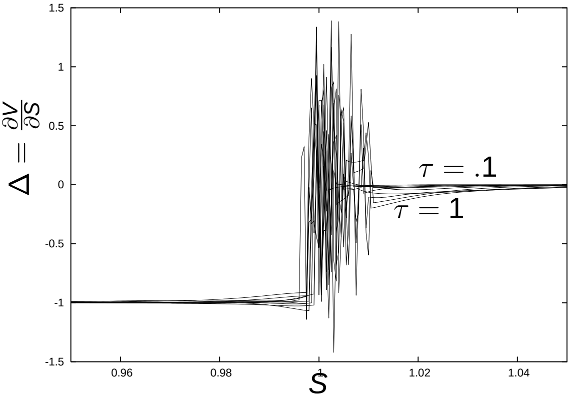
SIMULATION APPROACH

- Puts seem basically OK
- Difficulties with calls when

  $$(1 - \lambda V_{SS}^{BS})^2 \approx 0$$

  can cause $S >> 1$ and so $C >> 1$, but $P \approx 0$
- Could use (reliable) put values in P-C parity (still applicable) to evaluate call values
- No problem with PDE approach: zeroes of $(1 - \lambda V_{SS}^{BS})^2$ matched with zeroes of V_{SS}

FULL FEEDBACK - STANDARD MARCHING (CRANK-NICOLSON) RESULTS



Deltas for full-feedback (European) put, $X = 1$, $r = 0.04$, $\sigma = 0.2$, $\lambda = 0.1$, $T = 1$, $\tau = 0.1, 0.2, \ldots, 1$

LOCAL ANALYSIS CLOSE TO EXPIRY - FULL FEEDBACK

As noted earlier, $\tau \to 0$ is an intricate regime generally - consider this limit; appropriate scaling is given by

$$V = -\tau^{\frac{1}{2}}\eta\mathcal{H}(-\eta) + \tau\phi(\eta)$$

with $\eta = (S - X)/\sqrt{\tau}$ and $\mathcal{H}(\cdot)$ denoting the Heaviside function - necessary to recover the jump in the derivative across the strike - solution has a discontinuous delta (smooth solutions do not exist). The solution is described by

$$\phi - \frac{\eta}{2}\phi_\eta - \frac{\sigma^2 X^2 \phi_{\eta\eta}}{2\left(1 - \lambda\phi_{\eta\eta}\right)^2} + rX\mathcal{H}(-\eta) = 0,$$

subject to $\phi \to 0 \text{ as } \eta \to \infty, \quad \phi \to -rX \text{ as } \eta \to -\infty$. Turns out that there are two different solution regimes.

LOCAL ANALYSIS CLOSE TO EXPIRY (CONTINUED)

Re-write the nonlinear ODE (a quadratic in $\phi_{\eta\eta}$) in standard form to obtain

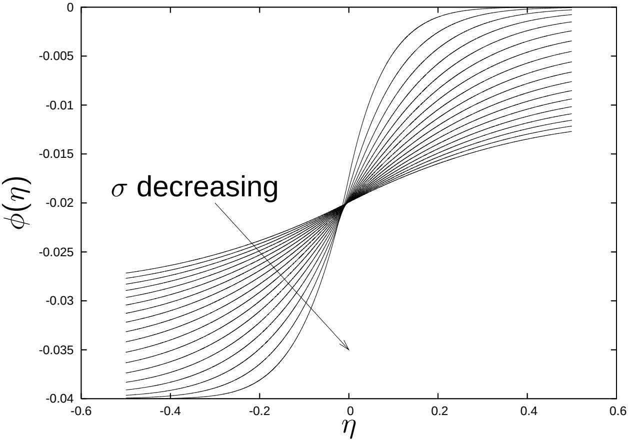
$$\phi_{\eta\eta} = \left(\frac{1}{\lambda} + \frac{\sigma^2 X^2}{4\lambda^2\psi}\right) - \frac{\sigma^2 X^2}{4\lambda^2\psi}\left(1 + \frac{8\lambda\psi}{\sigma^2 X^2}\right)^{\frac{1}{2}}$$

where $\psi = \phi - \frac{\eta}{2}\phi_\eta + rX\mathcal{H}(-\eta)$. Solution must undergo some qualitative change when/if the square-root becomes negative, i.e. when

$$1 + \frac{8\lambda\psi}{\sigma^2 X^2} < 0,$$

likely for sufficiently large λ or sufficiently small σ.

RESULTS - REGIME 1: $1 + 8\lambda\psi/(\sigma^2 X^2) > 0$



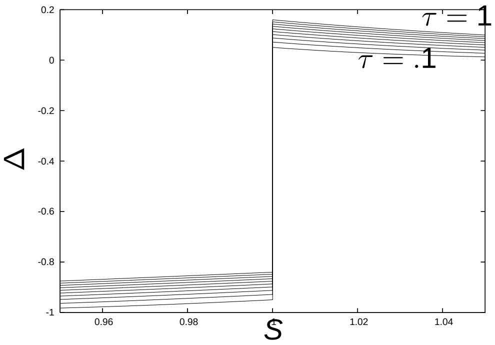
$X = 1$, $r = 0.04$, $\lambda = 0.1$ and $\sigma = 1, 0.95, \ldots, 0.15$.

MODIFIED TECHNIQUE FOR ‘FULL’ FULL-FEEDBACK PROBLEM

Guided by $\tau \to 0$ results, adopt following technique:-

- Scheme based on Keller (1978) technique
- System treated as 2 × first order in S (V, $V_1 = \frac{\partial V}{\partial S}$)
- At $S = X$, discontinuity in V_1 built in:
  - $V_1(S = X^+, \tau) = V_1(S = X^-, \tau) + 1$
- Combined with Crank-Nicolson marching in τ and Newton iteration

RESULTS FROM MODIFIED PROCEDURE



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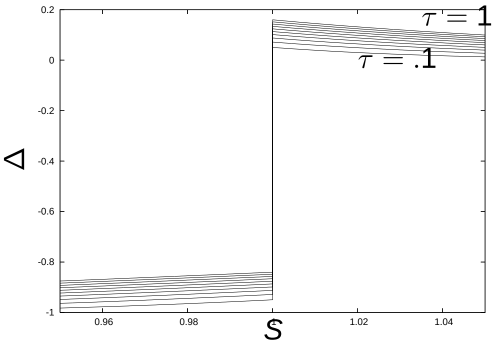

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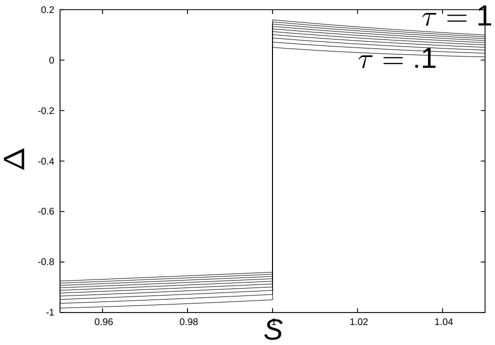

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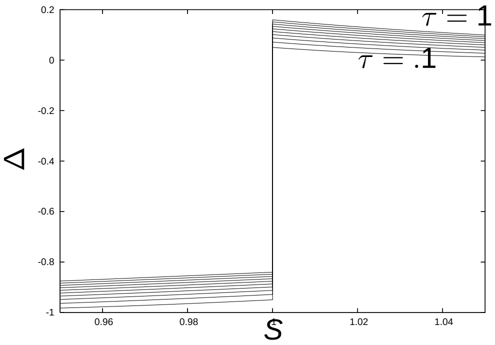

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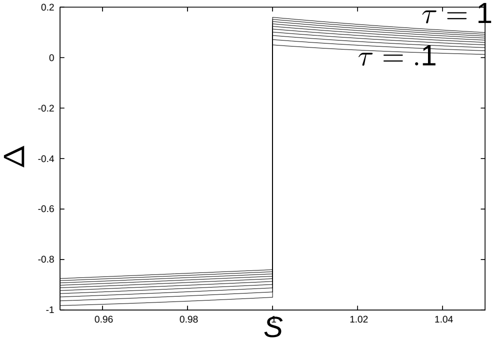
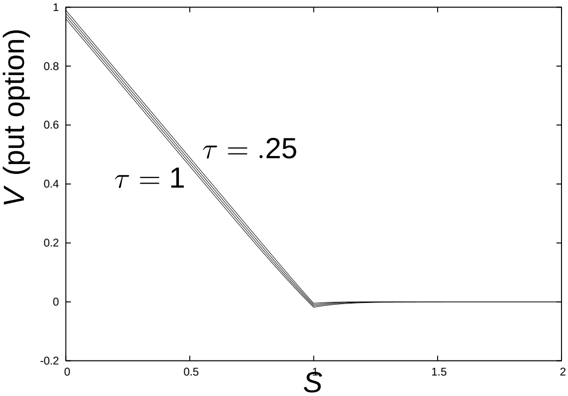


Full feedback put, $X = 1$, $r = 0.04$, $\sigma = 0.2$, $\lambda = 0.1$

Results indicate:

- Possibility of negative put values
- Never optimal to early exercise calls
- And....

ALSO...ANOTHER (NON-SMOOTH) SOLUTION TO THE BLACK-SCHOLES PROBLEM EXISTS

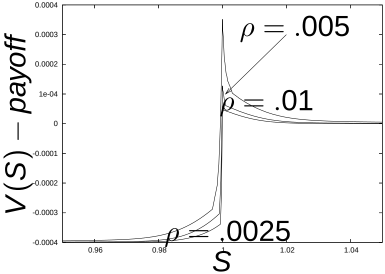


$X = 1$, $r = 0.04$, $\tau = T - t = 0.25, 0.5, 0.75, 1$, $\sigma = 0.2$, $\lambda = 0$

RESULTS - REGIME 2: $1 + 8\lambda\psi/(\sigma^2 X^2) < 0$

What happens when square-root term becomes negative?
Consider full (European) problem but with *smoothed payoff function* (original marching scheme) - an homotopy approach

$$V(S, \tau = 0) = \frac{1}{2}\left(X - S + \sqrt{(X - S)^2 + \rho^2}\right)$$



$X = 1$, $r = 0.04$, $\sigma = 1$, $\lambda = 0.1$ and $\tau = 0.1$.

IMPLICATIONS

Numerics suggest if $1 + \frac{8\lambda\psi}{\sigma^2 X^2} < 0$ the following (trivial) solution holds as $\rho \to 0$:

$$V(S,\tau) = \begin{cases} 0 & \text{for} \quad S > X, \\ Xe^{-r\tau} - S & \text{for} \quad S < X. \end{cases}$$

- Continuous option values
- Discontinuous deltas
- If early exercise permitted - ALWAYS exercise the put ($Xe^{-r\tau} - S < X - S \; \forall \tau > 0$)

THE CASE OF THE VANISHING DENOMINATOR - ANOTHER SUBTLETY

Turns out there can be problems even if the smoothing parameter ρ is finite:
The denominator in the PDE vanishes if

$$V_{SS}(S,\tau=0)=\frac{\rho^2}{2\left[(S-X)^2+\rho^2\right]^{\frac{3}{2}}}=\frac{1}{\lambda}$$

Gives critical locations:

$$S_0=X\pm\left[\left(\frac{\lambda\rho^2}{2}\right)^{\frac{2}{3}}-\rho^2\right]^{\frac{1}{2}}$$

Therefore critical points arise if $2\rho<\lambda$.
Hence to avoid the denominator vanishing have to impose an upper limit on the amount of elasticity (λ) or a lower limit on the smoothness parameter ρ.

THE CASE OF THE VANISHING DENOMINATOR

To further investigate what happens to the solution when the denominator vanishes consider local analysis around singular point S_0. Only appropriate solution form as $\tau \to 0$ is

$$V = V_0 + \tau^{\frac{1}{5}}\zeta V_1 + \tau^{\frac{2}{5}}\frac{\zeta^2}{2\lambda} + \tau^{\frac{3}{5}}\hat{V}(\zeta) + \dots$$

where

$$\zeta = \frac{S - S_0}{\tau^{\frac{1}{5}}}$$

and V_0 and V_1 known constants.

THE CASE OF THE VANISHING DENOMINATOR

Substitution into the PDE gives the ODE around the singularity as $\tau \to 0$ as

$$3\hat{V} - \zeta\hat{V}_\zeta - \frac{5\sigma^2 S_0^2}{2\lambda^3\hat{V}_{\zeta\zeta}} = 0,$$

with the boundary conditions (obtained from asymptotic matching)

$$\hat{V} \to A\zeta^3 \text{ as } |\zeta| \to \infty$$

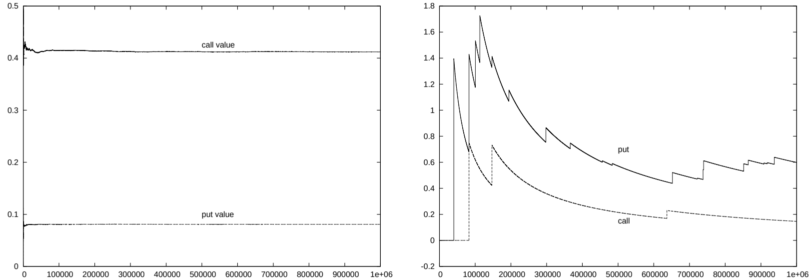
where A is a known constant.

It can be shown (using phase plane analysis) that this equation has no **smooth** solution even with smooth boundary (initial) conditions

SIMULATION METHODS - FULL FEEDBACK PROBLEM

- Must adopt an iterative approach
- Pick a set of random numbers (one for each timestep)
- Simulate three 'parallel' paths, starting at S_0, $S_0 + h$, $S_0 - h$ using same random numbers for each path
- Evaluate V_{SS} (numerical differentiation)
- Repeat simulations with V_{SS} from previous iteration, until converged.
- Many simulations fail to converge (discarded)

SIMULATION METHODS - FULL FEEDBACK RESULTS



(a) Put and call values (b) The gammas

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- SDE (simulation) methods generally hopeless!!