

# Can Time-Homogeneous Volatilities Produce any Call Prices?

Erik Ekström

Joint work with David Hobson and Johan Tysk.  
August 19, 2009, Stockholm

# The Black-Scholes Formula

Stock price (under  $Q$ ):

$$dX_t = rX_t dt + \sigma X_t dW$$

Call option price:

$$\begin{aligned} BS(x, t; K, T) &= E_{x,t} e^{-r(T-t)} (X_T - K)^+ \\ &= x\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \end{aligned}$$

**Note!** The BS-formula is often used in the other direction:  
Given an option price, there is a unique volatility  $\sigma$  that would produce this given price. This  $\sigma$  is called the 'implied volatility'.

# The Black-Scholes Equation

Local volatility model (under  $Q$ ):

$$dX_t = rX_t dt + \sigma(X_t, t)X_t dW$$

Call option price:

$$u(x, t; K, T) = E_{x,t} e^{-r(T-t)} (X_T - K)^+$$

The option price satisfies the Black-Scholes equation (in the current variables  $x, t$ !)

$$u_t + \frac{1}{2} \sigma^2(x, t) x^2 u_{xx} + rxu_x - ru = 0$$

with a terminal condition  $u(x, T; K, T) = (x - K)^+$ .

**Note!** The BS equation calculates the price of *one* option for *all* different current values  $x$  of the underlying.

# The Dupire Equation

In the forward variables  $(T, K)$ , the option price satisfies

$$u_T = \frac{1}{2} \sigma^2(K, T) K^2 u_{KK} - rK u_K$$

with initial condition  $u(x, t; K, t) = (x - K)^+$ .

**Note!** The Dupire equation calculates the price of *all* options for *one* fixed current value  $x$  of the underlying.

# Calibration of Models

In the Dupire equation, solve for the volatility:

$$\sigma^2(K, T) = 2 \frac{u_T + rKu_K}{K^2 u_{KK}}$$

Advantages of local volatilities:

- ▶ The model is rich enough so that any observed volatility surface can be fitted.
- ▶ There is an easy formula for the local volatility.

# Drawbacks of the Calibration of Local Vol Models

- ▶ The assumption of a double continuum of call prices is unrealistic.
- ▶ The solution is unstable with respect to small changes in the in-data.

# Recovery of a Time-Homogeneous Stock Price Process

Assume that call option prices  $C(K)$  are given for all strikes  $K$  but for a *fixed* maturity  $T$ . Also assume that  $C(K)$  is convex and decreasing,  $C(0) = x$ ,  $C(\infty) = 0$ .

**Question:** Can a time-independent local volatility  $\sigma(x)$  be found so that  $dX = \sigma(X) dB$  satisfies

$$C(K) = E_x(X_T - K)^+$$

for all  $K$ ?

# The Forward Problem and the Inverse Problem

**The forward problem:** Given  $\sigma(\cdot)$ , determine  $C(K)$ .

Solution: Solve the Dupire equation.

**The inverse problem:** Given  $C(K)$  for all  $K$ , determine  $\sigma(\cdot)$ .

Assume that  $X_T$  has a density

$$p(x, t; T, y) = P(X_T \in dy | X_t = x) / dy.$$

Then

$$C(K) = E_x(X_T - K)^+ = \int_0^\infty (y - K)^+ p(x, t; y, T) dy.$$

Differentiating twice gives

$$C''(K) = p(x, t; K, T).$$

# An Equivalent Problem

Given a probability distribution (centered around  $x$ ) and a deterministic time  $T$ , determine a time-homogeneous volatility  $\sigma(\cdot)$  so that

$$\begin{cases} dX_t = \sigma(X_t) dB \\ X_0 = x \end{cases}$$

has the given distribution at time  $T$ .

# Diffusions and the Kolmogorov Equations

Assume that  $X$  is a diffusion satisfying

$$dX_t = \sigma(X_t) dB_t.$$

Let  $p(x, t; y, T)$  be its density (assuming it exists).

The backward equation:

$$\begin{cases} p_t + \frac{1}{2}\sigma^2(x)p_{xx} = 0 \\ p(x, T; y, T) = \delta_y(x) \end{cases}$$

The forward equation:

$$\begin{cases} p_T = \frac{1}{2}(\sigma^2(y)p)_{yy} \\ p(x, t; y, t) = \delta_x(y) \end{cases}$$

**The forward problem:** Given  $\sigma(\cdot)$ , determine  $p(x, 0; y, T)$ .

Solution: Solve the forward equation.

**The inverse problem:** Given a probability distribution and a fixed time  $T$ , determine  $\sigma(\cdot)$  so that the distribution of  $X_T$  coincides with the given distribution.

# A Discretised Setting

Consider a process  $X$  that takes values in the finite set  $\{K_0, K_1, \dots, K_n, K_{n+1}\}$ . For simplicity, we assume that the  $K_i$ s are evenly spaced over  $[K_0, K_{n+1}]$ . Also assume that

- ▶  $K_0$  and  $K_{n+1}$  are absorbing states
- ▶  $X$  is skip-free, i.e. it can only jump to its two nearest neighbours
- ▶ the jump rate from  $K_i$  is  $b_i$ , and the probability of jumping up is the same as the probability of jumping down.

# Option Pricing

The price of an option with maturity  $T$  is given by

$$U(K_i, T) = E_{x_0}(X_T - K_i)^+.$$

For simplicity, however, we consider the function

$$u(K_i, t) = U(K_i, t) - x_0(K_{n+1} - K_i)/K_{n+1}.$$

Note that  $u(K_0, t) = u(K_{n+1}, t) = 0$  for all times  $t$ .

The pricing vector

$$u(T) := (u(K_1, T), \dots, u(K_n, T))^*$$

satisfies

$$u_T(T) = Au(T) \quad T > 0 \quad (1)$$

with initial condition  $u(0) = h := (u(K_1, 0), \dots, u(K_n, 0))^*$ . Here

$$A = \begin{pmatrix} -2b_1 & b_1 & 0 & \cdot & 0 \\ b_2 & -2b_2 & b_2 & \cdot & \\ & \cdot & \cdot & \cdot & 0 \\ & & \cdot & \cdot & b_{n-1} \\ & & 0 & b_n & -2b_n \end{pmatrix}$$

is the transition matrix ( $n \times n$ ) for the process  $X$ . The solution to (1) is given by

$$u(T) = e^{TA}u(0),$$

where  $e^{TA}$  is the matrix exponential of  $TA$ .

# The Forward Problem and the Inverse Problem

**The forward problem:** Given  $b$ , determine  $u(T)$ .

Solution:  $u(T) = e^{TA}u(0)$ .

**The inverse problem:** Given  $u(T)$ , determine  $b$ .

# Convexity and Monotonicity Results not to be Surprised by

Consider the forward problem, i.e. let the transition rates  $b = (b_1, \dots, b_n)$  be given.

## Proposition

*The option price  $u(T)$  is convex and lies between  $h$  and  $0$ .*

## Proposition

*The option price  $u(T)$  is increasing in the transition rates  $b = (b_1, \dots, b_n)$ .*

# Main Result (so far)

Let

$$\begin{aligned}\mathcal{C} &:= \{\text{convex vectors between } h \text{ and } 0\} \\ &= \{c \in R_+^n : (0, c, 0) \text{ is convex and } h_i \leq c_i \leq 0\}.\end{aligned}$$

## Theorem

*For any given vector  $u(T) \in \mathcal{C}$  there exists a unique solution to the inverse problem.*

## Sketch of Proof

Consider the function  $F : [0, \infty)^n \rightarrow \mathcal{C}$ . We want to show it is onto.

Note that  $\partial\mathcal{C}$  consists of faces of two different types:

1. Faces on which  $(u(T))_i = h_i$  for some  $i$ .
2. Faces on which the convexity of  $u(T)$  at some coordinate  $K_i$  is 0.

Transition rates  $b$  with  $b_i = 0$  for some  $i$  are mapped into the first type of boundary.

Transition rates  $b$  with  $b_i = \infty$  for some  $i$  are mapped into the second type of boundary.

Assume that these maps are onto as maps from an  $(n-1)$ -dimensional face  $\{b_i = 0\}$  (or  $\{b_i = \infty\}$ ) to the corresponding face of  $\mathcal{C}$ . Then also  $F$  is onto since the boundary is mapped onto the boundary and is a map of degree 1. Thus an induction argument gives that  $F$  is onto.

# To-Do List

- ▶ Uniqueness.
- ▶ Convergence to a "generalised diffusion" as the discretisation is made finer.
- ▶ Multiple maturities and convergence of the volatility to the Dupire volatility.
- ▶ Numerical schemes.

# To-Do List

- ▶ Uniqueness.
- ▶ Convergence to a "generalised diffusion" as the discretisation is made finer.
- ▶ Multiple maturities and convergence of the volatility to the Dupire volatility.
- ▶ Numerical schemes.

Thank you for your attention!