Switching Problems and Systems of PDEs with Inter-Connected Obstacles

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References

The talk is made on the basis of the following three papers.

- 1. B.Djehiche, S.H., A.Popier (2007): A Finite Horizon Optimal Multiple Switching Problem *(SIAM JCO)*
- 2. S.H., J.Zhang (2007): Switching Problem and Related System of Reflected Backward SDEs, *arXiv:0710.0908*
- 3. B.Elasri, S.H. (2009): The Finite Horizon Optimal Multi-Modes Switching Problem: the Viscosity Solution Approach, *Applied Mathematics and Optimization.*

Outlines

- Motivation and setting of the switching problem
- The Snell envelope notion
- Verification Theorem and its solution
- Connection with systems of reflected BSDEs
- The Markovian case and Systems of PDEs
- Knightian uncertainty
- Systems of reflected BSDEs with oblique reflection.

1. Motivation and Setting of the switching problem

- $B := (B_t)_{t \leq T}$ a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$.
- $(F_t)_{t \leq T}$ the completed natural filtration of B.
- **C** is a power plant.
- $X := (X_t)_{t \leq T}$ a stochastic process which stands for the price of electricity in the energy market or factors which determine the price of that commodity.

1.1. Features of the power plant

- The power plant has $m \geq 3$ modes of production (if e.g. $m = 3$, "1=no production", "2=normal mode" and "3=intensive one").
- Electricity cannot be stored, when produced it should be almost immediately consumed. Then the manager of **C** will put it dynamically in the most profitable mode.
- If **C** is in mode $i \in \mathcal{J} = \{1, ..., m\}$, the yield per dt is $\psi_i(t, \omega)dt$.
- Switching **C** from mode i to mode $j \neq i$ costs

$$\ell_{ij}(t, \omega) \geq \alpha > 0$$

  and continuous.
- ℓ_{ij} and ψ can be functions of X.

1.2. Admissible strategies of management

A management strategy of **C** has two components:

- 1. Let $\delta = (\tau_n)_{n\geq 0}$ be a sequence of stopping times such that $\tau_n \leq \tau_{n+1}$ and $\tau_n \to T$ P-*a.s.*. At τ_n the manager switches the production from the current mode to another one.
- 2. Let $\xi = (\xi_n)_{n\geq 0}$ be a sequence of $r.v.'s$ such that:

$$\xi_0 = 1 \text{ and } \forall n \geq 1, \xi_n(\omega) \in \mathcal{J} \text{ and } \xi_n \text{ } is \text{ } F_{\tau_n} - meas..$$

- The pair (δ, ξ) is called a strategy of management of the power plant. $\square$

1.3. The payoff

Let $(u_t)_{t\leq T}$ be the process indicator of the production mode at t of **[C]**:

$$u_0 = 1 \text{ and } u_t = \xi_n \text{ if } t \in]\tau_n, \tau_{n+1}] \ (n \geq 0).$$

When a strategy (δ, ξ) is implemented the yield is given by:

$$J(\delta, \xi) := E[\int_0^T \psi_{u_s}(s)ds - \sum_{n\geq 1} \ell_{u_{\tau_{n-1}}, u_{\tau_n}}(\tau_n) 1\!\!1_{[\tau_n < T]}].$$

1.4. Problems

1) Existence of an optimal strategy (δ^*, ξ^*), i.e.,

$$J(\delta^*, \xi^*) = \sup_{(\delta,\xi)} J(\delta, \xi).$$

2) What can be said about

$$\sup_{(\delta,\xi)} J(\delta, \xi)$$

in terms of characterization, properties, simulation,...□

Remark : $\sup_{(\delta,\xi)} J(\delta, \xi)$ is the price of the power plant in the energy market. The model fits also for a crude oil field of the North Sea or for tolling agreements. The states of the model could also be economies,... □

2. Snell envelope of processes

Let $U = (U_t)_{0 \leq t \leq T}$ be an F-adapted R-valued *rcll* process that belongs to class [D], *i.e.* the set $\{U_\tau, \tau \, stop.\, time\}$ is uniformly integrable. Then, there exists an F-adapted R-valued *rcll* process $Z := (Z_t)_{0 \leq t \leq T}$ such that:

- Z is the smallest *rcll* supermartingale of class [D] such that $Z \geq U$. It is called the *Snell envelope* of U.
- For any F-stopping time θ we have:

$$Z_\theta = esssup_{\tau \geq \theta} E[U_\tau | F_\theta] \quad (Z_T = U_T).$$

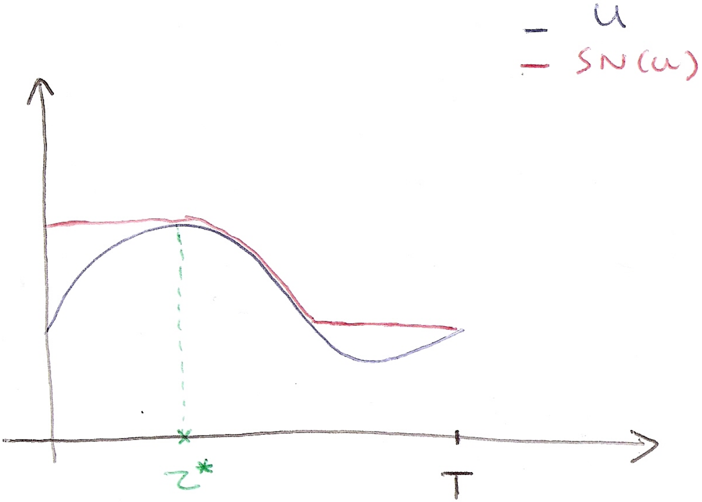

Figure: Continuous case

- Let U be continuous or only positive jumps. Then Z is a continuous process and if θ is a stopping time and

$$\tau_\theta^* = \inf\{s \geq \theta, Z_s = U_s\} \wedge T$$

then τ_θ^* is optimal after θ i.e.

$$E[U_{\tau^*}] = \sup_{\tau \geq \theta} E[U_\tau]. \ \Box$$

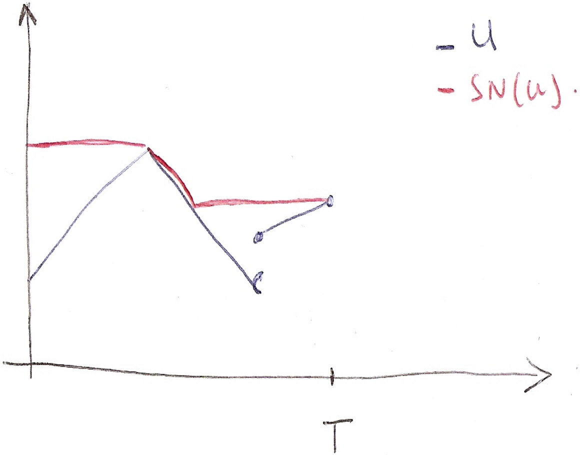

Figure: rcll with positive jumps

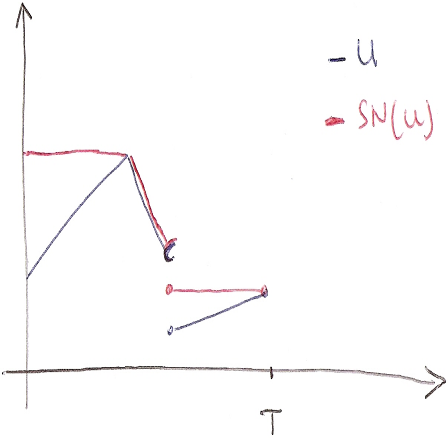

Figure: rcll with negative jumps

For $i=1,...,m$, set $\mathcal{J}^{-i}=\mathcal{J}-\{i\}$.

Theorem (Djehiche, H., Popier, 2007): There exists a unique m-uplet of continuous processes $(Y^1,\ldots,Y^m)$ such that:
for $i=1,\ldots,m$

$$\begin{cases} Y_t^i = esssup_{\tau\geq t} E[\int_t^\tau \psi_i(s)ds + \max_{j\in\mathcal{J}^{-i}}(-\ell_{ij}(\tau)+Y_\tau^j)1\!\!1_{[\tau<T]}|F_t]; \\ Y_T^i = 0. \end{cases}$$

- The processes Y^i, $i = 1, \ldots, m$, provide the optimal strategy $(\delta^*, \xi^*) = ((\tau_n^*)_{n\geq 1}, (\xi_n^*)_{n\geq 1})$ as follows:

$$\tau_1^* = \inf\{s \geq 0, Y_s^1 = \max_{j \in \mathcal{J}^{-1}} (-\ell_{1j}(s) + Y_s^j)\} \text{ and}$$

$$\xi_1^* = \text{argmax}_{j \in \mathcal{J}^{-1}} (-\ell_{1j}(\tau_1^*) + Y_{\tau_1^*}^j).$$

Next denote i_1 the maximum argument in

$$\max_{j \in \mathcal{J}^{-1}} (-\ell_{1j}(\tau_1^*) + Y_{\tau_1^*}^j)$$

then

$$\tau_2^* = \inf\{s \geq \tau_1^*, Y_s^{i_1} = \max_{j \in \mathcal{J}^{-i_1}} (-\ell_{i_1 j}(s) + Y_s^j)\}$$

and

$$\xi_2^* = \text{argmax}_{j \in \mathcal{J}^{-i_1}} (-\ell_{i_1 j}(\tau_2^*) + Y_{\tau_2^*}^j)\}$$

and so on.

- Finally it holds true that:

$$Y_0^1 = \sup_{(\delta,\xi)} J(\delta, \xi). \qquad \square$$

Proof.

- The system is the traduction of the Dynamical Programming Principle.
- The optimal stopping times are due to

$$Y_t^i + \int_0^t \psi_i(s)ds = \text{Snell Env.}\{\int_0^t \psi_i(s)ds$$

$$+ \max_{j \in \mathcal{J}^{-i}}(-\ell_{ij}(t) + Y_t^j)1\!\!1_{[t<T]}\}.$$

- $Y_0^1 = J(\delta^*, \xi^*)$ and $Y_0^1 \geq J(\delta, \xi)$, thus

$$Y_0^1 = \sup_{(\delta,\xi)} J(\delta, \xi).$$

- $Y_t^i :=$ the optimal yield of **C** if at t it is in mode i. Thus uniqueness.

□

- Proof of existence of $(Y^1, \ldots, Y^m)$: Approximations.
  For $i \in \mathcal{J}$,

$$Y_t^{i,0} = E[\int_t^T \psi_i(s)ds|F_t]$$

and, for $n \geq 1$,

$$Y_t^{i,n} = \text{ess sup}_{\tau \geq t} E[\int_t^\tau \psi_i(s)ds +$$

$$\max_{k \in \mathcal{J}^{-i}}(-\ell_{ik}(\tau) + Y_\tau^{k,n-1})1\!\!1_{[\tau < T]}|F_t].$$

Therefore:

- $Y^{i,n}$ is continuous (by induction)
- $Y_t^{i,n}$:= the optimal profit if at t **C** is in mode i and we allow for n switching at most
- $Y^{i,n} \leq Y^{i,n+1} \leq \zeta := E[\int_t^T \sum_{i=1,m} |\psi_i(s)|ds|F_t]$
- $Y_t^i := \lim_n Y_t^{i,n}$ is *rcll*
- Y^i, $i = 1, \ldots, m$, is a solution for the Verif. Theorem. □

4. Connection with systems of reflected BSDEs

Let $\xi \in L^2(F_T, P)$, $f(t, \omega, y, z)$ a function and $S := (S_t)_{t \leq T}$ an R-valued uniformly square integrable s.t. $S_T \leq \xi$. Assume F_t-adaptation. Then:

Theorem (El-Karoui et al., '97 or Ham., Popier '08 ($p < 2$)): There exists a unique triple $(Y_t, Z_t, K_t)_{t \leq T}$, valued in R^{1+d+1} and F_t-adapted (K continuous and increasing) such that :

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s, \ t \leq T; \\ Y_t \geq S_t \text{ and } \int_0^T (Y_t - S_t) dK_t = 0. \end{cases} \tag{1}$$

In addition, Y satisfies :

$$Y_t = \text{esssup}_{\tau \geq t} E\Big[\int_t^\tau f(s, \omega, Y_s, Z_s) ds + S_\tau 1\!\!1_{[\tau < T]} + \xi 1\!\!1_{[\tau = T]} \Big| \mathcal{F}_t\Big]. \square$$

Let $(t,x) \in [0,T] \times R^k$ and let $(X_s^{t,x})_{s\leq T}$ be the solution of the following standard SDE:

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(u, X_u^{t,x})du + \int_t^s \sigma(u, X_u^{t,x})dB_u, \;\; s \in [t,T] \\ X_s^{t,x} = x \text{ if } s \leq T. \end{cases}$$

Assume:

- $f(s,\omega,y,z) = f(s, X_s^{t,x}(\omega), y, z)$
- $\xi = g(X_T^{t,x})$
- $S_s = h(s, X_s^{t,x})$.

Then there exists a continuous deterministic function $v(t,x)$ such that for any $s \in [t,T]$, $Y_s = v(s, X_s^{t,x})$. Moreover v is a unique viscosity solution of:

$$\min\{v - h, -\partial_t v - \mathcal{L}v(t,x) - f(t,x,v(t,x),\sigma(t,x)D_x v(t,x)\} = 0;$$
$$v(T,x) = g(x),$$

where $\mathcal{L}$ is the infinitesimal generator of $X^{t,x}$. $\Box$

As a first consequence we have:

Theorem (Djehiche, H., Popier, '10): There exist unique triples $(Y^i, Z^i, K^i), i = 1, \ldots, m$, such that:

$$\begin{cases} Y_t^i = \int_t^T \psi_i(u)du - \int_t^T Z_u^i dB_u + K_T^i - K_t^i \\ Y_t^i \geq \max_{j \in \mathcal{J}^{-i}}\{-\ell_{ij}(t) + Y_t^j\}, \\ \int_0^T (Y_u^i - \max_{j \in \mathcal{J}^{-i}}\{-\ell_{ij}(u) + Y_u^j\})dK_u^i = 0. \end{cases} \tag{2}$$

□

5. The Markovian framework of switching: Connection with systems of PDEs

Recall $(X_s^{t,x})_{s\leq T}$ and suppose:

- $\psi_i(\omega, s) = \psi_i(s, X_s^{t,x}(\omega))$
- $\ell_{ij}(\omega, s) = \ell_{ij}(s, X_s^{t,x}(\omega))$

where $\psi_i(t,x)$ and $\ell_{ij}(t,x) \geq \alpha$ are deterministic continuous functions.

The HJB system associated with the switching problem is: $\forall i \in \mathcal{J}$,

$$\begin{cases} \min\{v_i(t,x) - \max_{j\in\mathcal{J}^{-i}}(-\ell_{ij}(t,x) + v_j(t,x)), \\ \qquad -\partial_t v_i(t,x) - \mathcal{L}v_i(t,x) - \psi_i(t,x)\} = 0, \\ v_i(T,x) = 0; \end{cases} \tag{3}$$

A solution in viscosity sense of this system is:

Definition

Let $(v_1, ..., v_m)$ be a m-uplet of continuous functions defined on $[0, T] \times R^k$, R-valued and such that $v_i(T, x) = 0$ for any $x \in R^k$ and $i \in \mathcal{J}$. The m-uplet $(v_1, ..., v_m)$ is called:

(i) a viscosity supersolution (resp. subsolution) of the system if for each fixed $i \in \mathcal{J}$, for any $(t_0, x_0) \in [0, T] \times R^k$ and any function $\varphi_i \in C^{1,2}([0, T] \times R^k)$ such that $\varphi_i(t_0, x_0) = v_i(t_0, x_0)$ and (t_0, x_0) is a local maximum of $\varphi_i - v_i$ (resp. minimum), we have:

$$\min\Big\{v_i(t_0, x_0) - \max_{j \in \mathcal{J}^{-i}}(-\ell_{ij}(t_0, x_0) + v_j(t_0, x_0)), \\ -\partial_t \varphi_i(t_0, x_0) - \mathcal{L}\varphi_i(t_0, x_0) - \psi_i(t_0, x_0)\Big\} \geq 0 \quad \text{(resp. } \leq 0).$$

(ii) a viscosity solution if it is both a viscosity supersolution and subsolution. □

Theorem (Elasri-Ham. '09): There exist deterministic continuous with polynomial growth functions $(v_i(t,x))_{i\in\mathcal{J}}$ such that:

- $Y_s^{i;t,x} = v_i(s, X_s^{t,x}), \forall s \in [t, T]$
- $(v_i(t,x))_{i\in\mathcal{J}}$ is the unique viscosity solution of the HJB system, in the class of continuous with polynomial growth functions. $\square$

Proof: Existence: Let $v^{i,n}$, $i \in \mathcal{J}$ and $n \geq 0$, be the deterministic functions such that:

$$\forall\, i \in \mathcal{J}, \forall s \in [t,x],\ \ Y_s^{i,n} = v^{i,n}(s, X_s^{t,x}).$$

Then:

- $v^{i,n}(t,x) \leq v^{i,n+1}(t,x) \leq E[\int_t^T \sum_{i\in\mathcal{J}} |\psi_i(s, X_s^{t,x})| ds] =: \Phi(t,x)$
- $v^i(t,x) := \lim_n v^i(t,x)$.

- The main difficulty is continuity of v_i because:

$$\begin{cases} v_i(s, X_s^{t,x}) = Y_s^{i;t,x} = \int_s^T \psi_i(s, X_s^{t,x})ds + \\ \qquad\qquad\qquad K_T^{i;t,x} - K_s^{i;t,x} - \int_s^T Z_s^{i;t,x} dB_s \\ Y_s^{i;t,x} \geq \max_{j \in \mathcal{J}^{-i}}(-\ell_{ij}(s, X_s^{t,x} + Y_s^{j;t,x}) \\ (Y_s^{i;t,x} - \max_{j \in \mathcal{J}^{-i}}(-\ell_{ij}(s, X_s^{t,x} + Y_s^{j;t,x}))dK_s^{i;t,x} = 0. \end{cases}$$

But since $\ell_{i,j}(t,x) \geq \alpha > 0$ then the optimal strategy $(\tau_n^*)_{n\geq 1}$ satisfies:

$$P[\tau_n^* < T] \leq C(1+|x|^p)/n.$$

Actually

$$Y_0^{1,tx} = E[\int_0^T \psi_{u_s^*}(s, X_s^{tx})ds - \sum_{n\geq 1} \ell_{u^*_{\tau^*_{n-1}} u^*_{\tau^*_n}}(\tau_n^*, X^{tx}_{\tau_n^*}) 1\!\!1_{[\tau_n^* < T]}]$$

Then split the sum up to n, use

$$\ell_{ij} \geq \alpha > 0 \text{ and } [\tau_n^* < T] \subset [\tau_k^* < T], \text{ for } k = 1, n.$$

We then reduce the set of admissible strategies to:

$$\tilde{D} := \{(\delta, \xi) = ((\tau_n)_{n\geq 1}, (\xi_n)_{n\geq 0}) \text{ admiss. such that}$$

$$\forall n \geq 1, P[\tau_n < T] \leq \frac{C(1+(1+|x|)^p)}{n}\}.$$

Therefore the strategy (δ^*, ξ^*) belongs to $\tilde{D}$ and:

$$y_0^{1,tx} = v_1(t,x) = \sup_{(\delta,\xi)\in\tilde{D}} E[\int_0^T \psi_{u_s}(s, X_s^{tx})1\!\!1_{[s\geq t]}ds$$

$$- \sum_{n\geq 1} \ell_{u_{\tau_{n-1}} u_{\tau_n}}(\tau_n \vee t, X_{\tau_n \vee t}^{tx})1\!\!1_{[\tau_n < T]}]$$

$$= sup_{(\delta,\xi)\in\tilde{D}} E[\int_0^{\tau_n} \psi_{u_s}(s, X_s^{tx})1\!\!1_{[s\geq t]}ds$$

$$- \sum_{1\leq k\leq n} \ell_{u_{\tau_{k-1}} u_{\tau_k}}(t \vee \tau_k, X_{t\vee\tau_k}^{tx})1\!\!1_{[\tau_k < T]} + 1\!\!1_{[\tau_n < T]} y_{\tau_n}^{u_{\tau_n},tx}].$$

Then:

$$|v_1(t,x) - v_1(t',x')| = |y_0^{1,tx} - y_0^{1,t'x'}| \leq sup_{(\delta,\xi)\in\tilde{D}} |(.....) - (....)|$$

and we deduce continuity.

Finally using the result by El-Karoui et al. we deduce that $(v^1, ..., v^m)$ is a solution of the HJB system. □

Uniqueness: If $u_1, ..., u_m$ and $w_1, ..., w_m$ are a subsolution and a supersolution of the system then for any $i = 1, ..., m$, $u_i \leq w_i$. For that we use the usual method of duplicating of variables. $\square$

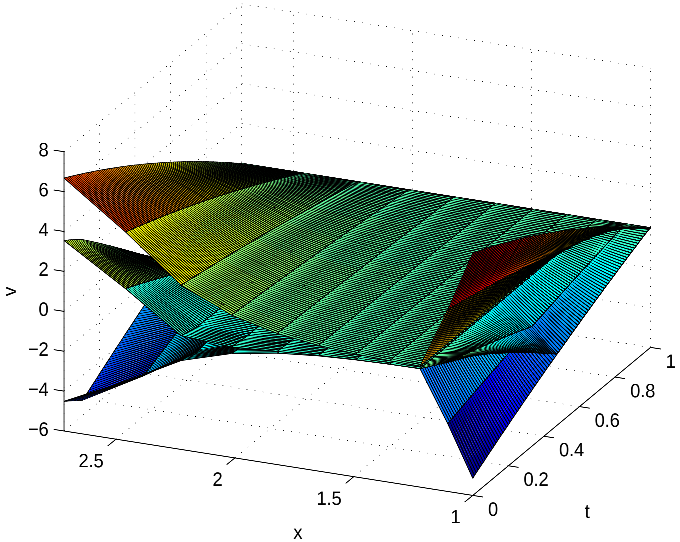

Figure: Three modes switching

6. The Switching Problem under Knightian uncertainty

Let X be the electricity price in the market and suppose that:

$$dX_t = X_t(r_t dt + \sigma dB_t),\ t \leq T \text{ and } X_0 = x > 0.$$

Then the price of the power plant is:

$$Sup_{(\delta,\xi)} J(\delta, \xi, X).$$

Suppose now that the probability of the future is not certain and a family of probabilities P^α are likewise. The proba.'s P^α are not far from P in the sense that P^α is equivalent to P and

$$\frac{dP^\alpha}{dP} = \exp\{\int_0^T \frac{\alpha_s}{\sigma} dB_s - \frac{1}{2}\int_0^T |\frac{\alpha_s}{\sigma}|^2 ds\}$$

with $\alpha_t \in [-\kappa, \kappa]$ (κ is the degree of ignorance).

Assume now that under P^α the price of electricity is given by:

$$dX_t = X_t(\alpha_t dt + \sigma dB^\alpha)$$

where $B_t^\alpha = B_t - \int_0^t \frac{\alpha_s}{\sigma} ds$.

Then the fair price of the power plant is given by:

$$J^* = \sup_{(\delta,\xi)} \inf_{\alpha} E^\alpha \Big[\int_0^T \psi_u(s, X_s) ds - \sum_{n \geq 1} \ell_{u_{\tau_{n-1}}, u_{\tau_n}}(\tau_n) 1\!\!1_{[\tau_n < T]} \Big]$$

where E^α is the expectation under P^α.

Modelisation: The case $m = 2$. Let X be such that

$$dX_t = \sigma X_t dB_t, X_0 = x_0$$

and

$$H^*(t,z) = inf_{\alpha \in [-\kappa,\kappa]} \frac{z\alpha}{\sigma}.$$

Let the following system:

$$\begin{cases} Y_t^1 = \int_t^T [\psi_1(s,X_s) + H^*(s,Z_s^1)]ds - \int_t^T Z_s^1 dB_s + K_T^1 - K_t^1; \\ Y_t^2 = \int_t^T [\psi_2(s,X_s) + H^*(s,Z_s^2)]ds - \int_t^T Z_s^2 dB_s + K_T^2 - K_t^2; \\ Y_t^1 \geq Y_t^2 - \ell_{12}(t); \quad [Y_t^1 - Y_t^2 + \ell_{12}(t)]dK_t^1 = 0; \\ \\ Y_t^2 \geq Y_t^1 - \ell_{21}(t); \quad [Y_t^2 - Y_t^1 + \ell_{21}(t)]dK_t^2 = 0. \end{cases}$$

Theorem (Ham.-Zhang, '07): If this system of reflected BSDEs with inter-connected obstacles has a solution then it is unique and

$$Y_0^1 = \sup_\delta \inf_\alpha J(\delta,\alpha).$$

Morever there exist δ^* and u^* such that $Y_0^1 = J(\delta^*, u^*)$. $\square$

7. Systems of reflected BSDEs with oblique reflection.

More generally we can consider the following system of reflected BSDEs with oblique reflection. For $i = 1, ..., m$,

$$\begin{cases} Y_t^i = \xi_i + \int_t^T f_i(u, Y_u^1, ..., Y_u^m, Z_u^i)du - \int_t^T Z_u^i dB_u + K_T^i - K_t^i \\ Y_t^i \geq \max_{j \in \mathcal{J}^{-i}} h_{ij}(\omega, t, Y_t^j) \\ \int_0^T (Y_u^i - \max_{j \in \mathcal{J}^{-i}} h_{ij}(\omega, u, Y_u^j))dK_u^i = 0. \end{cases} \quad (4)$$

Assumptions [H1]

For any $j = 1, ..., m$, it holds that:

- $E\left\{ \int_0^T \sup_{\overrightarrow{y}: y_j = 0} |f_j(t, \overrightarrow{y}, 0)|^2 dt + |\xi_j|^2 \right\} < \infty.$
- $f_j(t, \overrightarrow{y}, z)$ is uniformly Lipschitz continuous in (y_j, z) and is continuous in y_i for any $i \neq j$; and $h_{i,j}(t, y)$ is continuous in (t, y) for $j \in \mathcal{J}^{-i}$,
- $f_j(t, \overrightarrow{y}, z)$ is increasing in y_i for $i \neq j$, and $h_{i,j}(t, y)$ is increasing in y for $j \in \mathcal{J}^{-i}$.
- For $j \in \mathcal{J}^{-i}$, $h_{i,j}(t, y) \leq y$. Moreover, there is no sequence $j_2 \in \mathcal{J}^{-j_1}, \cdots, j_k \in \mathcal{J}^{-j_{k-1}}, j_1 \in \mathcal{J}^{-j_k}$, and $(y_1, \cdots, y_k)$ such that

$$y_1 \stackrel{\triangle}{=} h_{j_1,j_2}(t, y_2),\ y_2 \stackrel{\triangle}{=} h_{j_2,j_3}(t, y_3),\ \cdots,\ y_{k-1} \stackrel{\triangle}{=} h_{j_{k-1},j_k}(t, y_k),$$
$$y_k \stackrel{\triangle}{=} h_{j_k,j_1}(t, y_1).$$

- For any $i = 1, ..., m$, $\xi_i \geq \max_{j \in \mathcal{J}^{-i}} h_{i,j}(T, \xi_j)$.

Remark: Those assumptions on $h_{ij}(t,y)$ are satisfied if $h_{ij}(t,\omega,y) = y - \ell_{ij}(t,\omega)$ with $\ell_{ij}(t,\omega) > 0$.

Theorem (Ham.- Zhang '07): Under [H1] the system of reflected BSDEs (4) has a minimal solution.

Idea of the proof: Picard iterations.

$$\begin{cases} Y_t^{i,n} = \xi_i - \int_t^T Z_s^{i,n} dB_s + K_T^{i,n} - K_t^{i,n} + \\ \int_t^T f_i(s, Y_s^{1,n-1}, \cdots, Y_s^{i-1,n-1}, Y_s^{i,n}, Y_s^{i+1,n-1} \cdots, Y_s^{m,n-1}, Z_s^{i,n}) ds; \\ Y_t^{i,n} \geq \max_{j \in \mathcal{J}^{-i}} h_{i,j}(t, Y_t^{j,n-1}); \quad [Y_t^{i,n} - \max_{j \in \mathcal{J}^{-i}} h_{i,j}(t, Y_t^{i,n-1})] dK_t^{i,n} = 0. \end{cases}$$

with

$$Y_t^{i,0} = \xi_i + \int_t^T \inf_{\overrightarrow{y} : y_j = y} f_i(t, \overrightarrow{y}, z)(s, Y_s^{i,0}, Z_s^{i,0}) ds - \int_t^T Z_s^{i,0} dB_s, i = 1, m.$$

Then comparison implies:

- $Y^{i,n} \leq Y^{i,n+1}$
- $Y^{i,n} \leq \hat{Y}$

for some $(\hat{Y}, \hat{Z})$ solution of an appropriate BSDE.

- By Peng's monotonic theorem ('99)

$$Y_t^i = \lim_n Y_t^{i,n}$$

is a rcll process and

$$(Z^{i,n})_n \to Z^i \text{ in } L^p(dt \otimes dP) \ (p \in [1, 2[).$$

Let $K_t^i = \lim_n K_t^{i,n}$, then:

$$\begin{cases} Y_t^i = \xi_i + \int_t^T f_i(s, \overrightarrow{Y}_s, Z_s^i)ds - \int_t^T Z_s^i dB_s + K_T^i - K_t^i; \\ Y_t^i \geq \max_{j \in \mathcal{J}^{-i}} h_{i,j}(t, Y_t^j). \end{cases}$$

- Consider now the following RBSDEs:

$$\begin{cases} \tilde{Y}_t^i = \xi_i - \int_t^T \tilde{Z}_s^i dB_s + \tilde{K}_T^i - \tilde{K}_t^i \\ \qquad\qquad + \int_t^T f_i(s, Y_s^1, \cdots, Y_s^{i-1}, \tilde{Y}_s^i, Y_s^{i+1}, \cdots, Y_s^m, \tilde{Z}_s^i) ds; \\ \tilde{Y}_t^i \geq \max_{j \in \mathcal{J}^{-i}} h_{i,j}(t, Y_t^j); \quad [\tilde{Y}_{t-}^i - \max_{j \in \mathcal{J}^{-i}} h_{i,j}(t, Y_{t-}^j)] d\tilde{K}_t^i = 0. \end{cases}$$

$\tilde{Y}^i$ is the smallest f_i-supermartingale with lower barrier $\max_{j \in \mathcal{J}^{-i}} h_{i,j}(t, Y_t^j)$ (Peng-Xu '05), then

$$\tilde{Y}_t^i \leq Y_t^i.$$

But $Y_t^{i,n-1} \leq Y_t^i$ for any $(i, n-1)$ then comparison implies that $Y_t^{i,n} \leq \tilde{Y}_t^i$ and then $Y^i \leq \tilde{Y}^i$, thus $\tilde{Y}^i = Y^i$.

Hence (Y^i, Z^i, K^i) satisfies a system of reflected BSDEs. Finally the Y^i's are continuous thanks to Assumption (4) on h_{ij}.

Remarks: • The solution is minimal.

• If f_i and h_{ij} are Lipschitz, the solution is unique.

• We can also make a link with systems of PDEs with inter-connected obstacles. □