Decision making under time inconsistency

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Time inconsistency

- Real life phenomena: Procrastination and addiction
- It is of economic relevance when the current action depend on the expectation of future actions.
- Examples: Government fiscal policy, Large shareholder ownership policy, overlapping generations etc...
- Optimal control becomes meaningless

Behavior under time inconsistency

- We focus on time inconsistency due to non constant discounting:
  - Hyperbolic discount
  - Utilitarian government in an overlapping generations model
- The behavior is the equilibrium of a game
- Additional assumptions:
  - Naivete versus sophistication
  - availability/absence of Commitment mechanisms
- We model the idea of instantaneous self

Results

- We characterize the equilibrium with a "differential equation" that has a non local term (the strategic term)
- The equation becomes the HJB equation when the discount rate is constant
- Few results and many open mathematical questions
- For special discounts function, the equation becomes a system of two ODEs with multiple solutions

Outline of the presentation

- The canonical DU model and the HJB equation
- The "instantaneous self" and the equilibrium
- Equilibrium characterization
- Special discount and long term behavior

I. The canonical DU model and the HJB equation

The canonical model of discounted utility (DU)

$$V(k) = \sup_{c} \int_{t}^{\infty} e^{-\delta(s-t)} u(c(s)) ds$$

under the state equation

$$\frac{dk(s)}{ds} = f\left(k(s)\right) - c(s), \qquad k(t) = k$$

where k is the capital stock, c is consumption and f is the production function, u is the utility function and δ is the constant discount rate.

Preference alignment of the DU model

- Fix a consumption path c^0 and a date $t > 0$

$$\overbrace{\int_0^\infty e^{-\delta s} u(c^0(s)) ds}^{\text{self "0" interest}} =$$

$$\int_0^t e^{-\delta s} u(c^0(s)) ds + e^{-\delta t} \underbrace{\int_t^\infty e^{-\delta(s-t)} u\left(c^0(s)\right) ds}_{\text{self "t" interest}}$$

- Due to exponential discount, self 0 and self t interest are aligned: future preferences confirm earlier preferences.

Solution method

Exploiting the stationarity of DU at the infinitesimal level, allows the derivation of the optimal consumption policy ($\overline{c}(k)$) through HJB

$$\delta V(k) = \sup_{c} \left[u(c) + V'(k) \left(f(k) - c \right) \right]$$

and the optimal consumption is then

$$u' \left(\overline{c}(k) \right) = V'(k) \quad \text{where } i = u'^{-1}.$$

II. The “instantaneous self” and the equilibrium

Non exponential discounting

$$V(k) = \text{``}\sup_{c}\text{''} \int_{t}^{\infty} h(s-t)u(c(s))ds$$

under the state equation

$$\frac{dk(s)}{ds} = f\left(k(s)\right) - c(s), \qquad k(t) = k.$$

- h is a decreasing positive function with $h(0) = 1$.
- Unless $h'/h =$ constant, preferences switch with the mere passage of time and Hamilton Jacobi solutions are not "implementable".

Policies

- We consider Markov-stationary and continuously differentiable consumption policy $\sigma : R \longrightarrow R, \quad k \mapsto \sigma(k)$
- The policy σ converges to $\bar{k}$ (the steady state) if the flow

$$\frac{dk}{ds} = f\left(k\left(s\right)\right) - \sigma\left(k\left(s\right)\right),\ k\left(0\right) = k_0$$

  if $\lim_{s \longrightarrow \infty} k\left(s\right) = \bar{k}$, when the initial value k_0 is sufficiently close to $\bar{k}$.

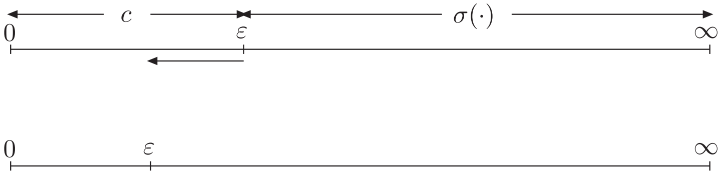
Equilibrium without commitment

- Assume that self 0 can dictate a given consumption level c for all selves on the interval $[0, \varepsilon]$ for some $\varepsilon > 0$.
- The expectation of self 0 is that all instantaneous selves on $(\varepsilon, \infty]$ will use the policy σ.
- Given this expectation, Self t contemplates the best consumption level c she can pick (for her coalition): call it $c(k, \varepsilon)$
- The policy σ is an *equilibrium* if no self k by himself has an incentive to deviate from $\sigma(\cdot)$ when the commanded coalition is vanishingly small.
- Mathematically

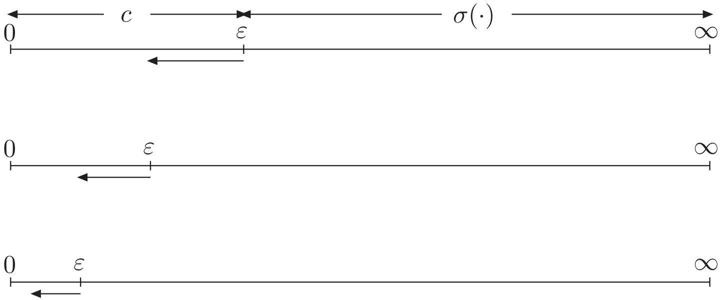
$$\lim_{\varepsilon \downarrow 0} c(k, \varepsilon) = \sigma(k).$$

The game structure

The game structure



The game structure



III Equilibrium characterization

Characterization of the equilibrium

Theorem: A C^1 consumption policy $\sigma : R \to R$ converging to $\bar{k}$ is an equilibrium if and only if there exist a function v satisfying the *differentiated equation* (DE)

$$-\int_0^\infty h'(t) u \circ i \left(v'(k_0(t)) \right) dt = \sup_c \left[u(c) + v'(k) \left(f(k) - c \right) \right] \tag{DE}$$

where

$$\frac{dk_0}{ds} = f\left(k_0\left(t\right)\right) - i\left(V'(k_0(t))\right), \quad k_0\left(0\right) \; = k$$

with the boundary condition

$$v\left(\bar{k}\right) = u\left(f\left(\bar{k}\right)\right) \int_0^\infty h\left(t\right) dt \tag{BC}$$

and such that

$$u'\left(\sigma(k)\right) = V'(k).$$

Remark: With exponential discount $h'(t) = -\delta h(t)$, the left hand side of DE becomes $\delta V(k)$ and DE becomes the standard HJB equation

Equilibrium policies

- Despite the preference reversal, the equilibrium policy σ will follow through in the absence of commitment: The equilibrium policy is time consistent.
- σ is a "resignation" policy (second best): Self t will be better off if he could command the decisions of the future selves.
- Typically, the optimal committed policy (which obeys HJB) is not an equilibrium.

Strategic nature of the non local term

- Suppose $h(t) = e^{-\delta_0 t}$ for $t \leq \tau$ and $h(t) = e^{-\delta_1 t}$ for $t > \tau$, with $\tau > 0$ and $\delta > \rho$.
- DE becomes

$$\delta_0 v(k) = \sup_c \left[u(c) + (\delta_0 - \delta_1)\, g(k) + V'(k)\,(f(k) - c) \right]$$

- The "naive" behavior:

$$\delta_0 V(k) = \sup_c \left[u(c) + V'(k)\,(f(k) - c) \right]$$

- Assuming linear technology technology $f(k) = Ak$ and the utility $u(c) = \log(c)$ we find

$$\sigma_{\mathrm{n}}(k) = \delta_0 k, \quad \sigma_e(k) = \frac{\delta_0}{1 + \frac{\delta_0 - \delta_1}{\delta_1} e^{-\delta_1 \tau}} k.$$

IV Special discount

Special discount: mixture of exponential

- The objective is

$$\int_0^\infty \left(\frac{\delta - \rho}{\pi + \delta - \rho} e^{-(\delta + p)t} + \frac{\pi}{\pi + \delta - \rho} e^{-\rho t} \right) \log(C(t)) dt$$

  with $\delta > \rho$.
- Hyperbolic discount: with some choice of (δ, π, ρ) we have $-\frac{h'}{h}$ is a decreasing function of time.
- Central planner in an overlapping economy where π is the intensity of death, δ is the individual discount and ρ is the planners discount rate.

Special discount: mixture of exponential

- With the mixture of exponentials, the equation DE becomes a system of two ODEs

$$\left(f - \frac{1}{v'}\right) v' - \ln\left(v'\right) = \delta v - (\delta - \rho)\, w, \tag{1}$$

$$\left(f - \frac{1}{v'}\right) w' = -\pi v + (\rho + \pi)\, w \tag{2}$$

with the boundary conditions

$$v\left(\bar{k}\right) = \frac{\rho + \pi}{\rho\,(\delta + \pi)} \ln f\left(\bar{k}\right) \tag{3}$$

$$w\left(\bar{k}\right) = \frac{\pi}{\rho\,(\delta + \pi)} \ln f\left(\bar{k}\right) \tag{4}$$

and the strategy σ is given by $\sigma(k) = 1/v'(k)$.

Existence of multiple equilibria

Theorem: Consider any $\bar{k} \in I$ where $I = \{k \mid \rho\frac{\pi+\delta}{\pi+\rho} < f'(k) < \delta\}$. Then there exist an equilibrium strategy, defined on some neighborhood Ω of $\bar{k}$ and converging to $\bar{k}$.

Proof:

- Change of variables and coordinate to reduce (1)-(2) to a three dimensional sysytem system such that, when linearized, it has two eigenvalues 0 and $\delta - f'(\bar{k})$
- We are then in a position to apply the central manifold theorem for existence of a central manifold
- Stability requires that $\bar{k}$ should belong to the interval I.

Conclusion.

- We provide the equivalent of the HJB equation for the policy games with non commitment (subgame perfect equilibria): This is the DE equation.
- Open mathematical issues with DE: existence, multiple solutions, long term qualitative behavior
- With particular discount, we could prove existence of multiple equilibria
- Multiple equilibria are attractive from a policy perspective because it gives a role for government intervention