

Optimal Control of Forward-Backward Stochastic Differential Equations and Risk Minimization

Bernt Øksendal

Center of Mathematics for Applications (CMA)
University of Oslo, Norway

oksendal@math.uio.no

Conference on PDE and Mathematical Finance III
KTH Stockholm, 17- 20 August 2009

Joint work with Agnès Sulem (INRIA, Paris-Rocquencourt)



Outline

- ▶ Motivation: risk minimizing portfolio problem
- ▶ Maximum principles for optimal control of FBSDE driven by Lévy processes
 - ▶ a sufficient maximum principle
 - ▶ an equivalence principle
 - ▶ a Malliavin calculus approach
- ▶ Application to risk minimizing portfolios



Financial market set up

Filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$.

- A *risk free asset*, with unit price $S_0(t) = 1$ for all $t \in [0, T]$
- A *risky asset*, with unit price $S(t)$

$$dS(t) = S(t^-)[\mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t, z)\tilde{N}(dt, dz)]; S(0) > 0$$

- $B(t)$: \mathcal{F}_t -Brownian motion
- $\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$: *compensated* measure of the *jump measure* $N(\cdot, \cdot)$ of a Lévy process $\eta(\cdot)$, ν being the *Lévy measure* of $\eta(\cdot)$.
- $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$
- $\mu(t), \sigma(t)$ and $\gamma(t, z)$: \mathcal{F}_t -predictable processes s.t. $\gamma(t, z) \geq -1 + \epsilon$ and

$$\int_0^T \left\{ |\mu(t)| + \sigma^2(t) + \int_{\mathbb{R}_0} \gamma^2(t, z)\nu(dz) \right\} dt < \infty \quad \text{a.s.}$$

Risk minimizing portfolio problem

The wealth process A_u corresponding to a portfolio u is given by

$$\begin{cases} dA(t) = A(t^-)u(t) [\mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t, z)\tilde{N}(dt, dz)] \\ A(0) = a > 0. \end{cases} \quad (1)$$

Pb: find $u^* \in \mathcal{A}_{\mathcal{E}}$ which minimizes the risk of the terminal wealth, i.e.

$$\inf_{u \in \mathcal{A}_{\mathcal{E}}} \rho(A_u(T)) = \rho(A_{u^*}(T))$$

where ρ is a *convex risk measure*, i.e. a map satisfying convexity, monotonicity and translation properties.

Definition of convex risk measures

Let $\mathbb{F} = L^p(\mathcal{F}_T, P)$ ($p \in [1, \infty]$) represent the family of all financial standings at time T . A *convex risk measure* is a map

$$\rho : \mathbb{F} \rightarrow \mathbb{R}$$

satisfying the following conditions

$$\text{(convexity)} \quad \rho(\lambda F + (1 - \lambda)G) \leq \lambda\rho(F) + (1 - \lambda)\rho(G) \text{ for all, } F, G \in \mathbb{F}, \quad (2)$$

$$\text{(monotonicity)} \quad F \leq G \Rightarrow \rho(G) \leq \rho(F); F, G \in \mathbb{F}. \quad (3)$$

(See e.g. Frittelli and Rosazza-Gianin (2002), Föllmer and Schied (2002) and Biagini and Frittelli (2007)).

Intuitively, $\rho(F)$ is the amount that has to be added to F to make the financial standing “acceptable”.

A representation of convex risk measures

There are several useful representations of convex risk measures. One of them is the following:

Theorem (Frittelli and Rosazza-Gianin, Föllmer and Schied)

Every convex risk measure $\rho : \mathbb{F} \rightarrow \mathbb{R}$ which also satisfies the translation invarianec property

$$\rho(F + a) = \rho(F) - a ; \forall F \in \mathbb{F}, a \in \mathbb{R} \quad (4)$$

is of the form

$$\rho(F) = \sup_{Q \in \mathcal{P}} \{E_Q[-F] - \zeta(Q)\} ; F \in \mathbb{F} \quad (5)$$

for some family \mathcal{P} of probability measures absolutely continuous with respect to P and some convex “penalty” function $\zeta : \mathcal{P} \rightarrow \mathbb{R}$.

For example, the *entropic risk measure* is defined by:

$$\rho(F) := \sup_{Q \ll P} \{E_Q[-F] - H(Q, P)\}$$

where H is the *relative entropy*

$$H(Q, P) = E \left[\frac{dQ}{dP} \ln \left(\frac{dQ}{dP} \right) \right]$$

With the representation (5), the problem of *minimizing the risk of the terminal wealth* leads to a *stochastic differential game*.

(Mataramvura-Øksendal)



Representation of risk measures by BSDEs (Peng, ...)

Definition: Define the *risk* $\rho_g(F)$ (associated to a convex function g) of a financial position F as

$$\rho_g(F) := \mathcal{E}_g[-F] := X_g^{-F}(0) \in \mathbb{R} \quad (6)$$

where $X_g^{-F}(0)$ is the value at $t = 0$ of the solution $X(t)$ of the BSDE:

$$\begin{cases} dX(t) = -g(X(t), Y(t), K(t, \cdot))dt + Y(t)dB(t) + \int_{\mathbb{R}_0} K(t, z)\tilde{N}(dt, dz) \\ X(T) = -F. \end{cases}$$

Remark: When $g(x, y, k) = \frac{1}{2}y^2$ and $\tilde{N}(dt, dz) = 0$, then ρ_g coincides with the *entropic risk measure* (Barrieu and El Karoui).



Now, the risk minimizing portfolio problem

$$\inf_{u \in \mathcal{A}} \rho_g(A_u(T))$$

is equivalent to

$$\inf_{u \in \mathcal{A}} X_g^{-A_u(T)}(0) \quad (7)$$

where $X_g^{-A_u(T)}(t)$ is given by the BSDE

$$\begin{cases} dX(t) = -g(X(t), Y(t), K(t, \cdot))dt + Y(t)dB(t) + \int_{\mathbb{R}_0} K(t, z)\tilde{N}(dt, dz) \\ X(T) = -A_u(T). \end{cases}$$

and $A(t)$ is given by a SDE.

This is an example of a *stochastic control problem of a system of FBSDEs driven by Lévy processes*.



Optimal control with partial information of FBSDEs

Forward system in the unknown process $A(t)$

$$\begin{cases} dA(t) = b(t, A(t), u(t))dt + \sigma(t, A(t), u(t))dB(t) \\ \quad + \int_{\mathbb{R}} \gamma(t, A(t), u(t), z)\tilde{N}(dt, dz); \quad t \in [0, T] \\ A(0) = a \in \mathbb{R} \end{cases} \quad (8)$$

Backward system in the unknown processes $X(t), Y(t), K(t, z)$

$$\begin{cases} dX(t) = -g(t, A(t), X(t), Y(t), u(t))dt + Y(t)dB(t) \\ \quad + \int_{\mathbb{R}} K(t, z)\tilde{N}(dt, dz); \quad t \in [0, T] \\ X(T) = cA(T), \quad c \in \mathbb{R} \setminus \{0\} \end{cases} \quad (9)$$

Admissible controls

Consider a subfiltration $\mathcal{E}_t \subseteq \mathcal{F}_t$ representing the information available to the controller at time t , e.g.

$$\mathcal{E}_t = \mathcal{F}_{(t-\delta)^+} \quad (\delta > 0 \text{ constant})$$

i.e. the controller gets a *delayed* information flow

- Let $\mathcal{A}_{\mathcal{E}}$ denote the family of admissible controls, contained in the set of \mathcal{E}_t -predictable controls $u(\cdot)$ such that the system (8)–(9) has a unique strong solution.
- U : given convex set s.t. $u(t) \in U, \forall t \in [0, T]$

Optimal control problem

Performance functional:

$$J(u) = E \left[\int_0^T f(t, A(t), X(t), Y(t), K(t, \cdot), u(t)) dt + h_1(X(0)) + h_2(A(T)) \right]; \quad u \in \mathcal{A}_{\mathcal{E}} \quad (10)$$

where f, h_1, h_2 are given functions s.t.

$$E \left[\int_0^T |f(t, A(t), X(t), Y(t), K(t, \cdot), u(t))| dt + |h_1(X(0))| + |h_2(A(T))| \right] < \infty.$$

Find $\Phi_{\mathcal{E}} \in \mathbb{R}$ and $u^* \in \mathcal{A}_{\mathcal{E}}$ such that

$$\Phi_{\mathcal{E}} = \sup_{u \in \mathcal{A}_{\mathcal{E}}} J(u) = J(u^*) \quad (11)$$

Hamiltonian

The Hamiltonian is defined by

$$\begin{aligned} H(t, a, x, y, k, u, \lambda, p, q, r) & \qquad (12) \\ &= f(t, a, x, y, k, u) + g(t, a, x, y, u)\lambda + b(t, a, u)p \\ &+ \sigma(t, a, u)q + \int_{\mathbb{R}_0} \gamma(t, a, u, z)r(z)\nu(dz) \end{aligned}$$

We assume that H is Frechet differentiable (C^1) in the variables a, x, y, k .

Pair of FBSDEs in the adjoint processes

Forward system in the unknown process $\lambda(t)$

$$\begin{cases} d\lambda(t) = \frac{\partial H}{\partial x}(t, A(t), X(t), Y(t), K(t, \cdot), u(t), \lambda(t), p(t), q(t), r(t, \cdot))dt \\ \quad + \frac{\partial H}{\partial y}(\cdot)dB(t) + \int_{\mathbb{R}_0} \nabla_k H(\cdot) \tilde{N}(dt, dz) \\ \lambda(0) = h'_1(X(0)) \quad (= \frac{dh_1}{dx}(X(0))) \end{cases} \quad (13)$$

Backward system in the unknown processes $p(t), q(t), r(t, \cdot)$

$$\begin{cases} dp(t) = -\frac{\partial H}{\partial a}(\cdot)dt + q(t)dB(t) + \int_{\mathbb{R}} r(t, z) \tilde{N}(dt, dz); \quad t \in [0, T] \\ p(T) = c\lambda(T) + h'_2(A(T)) \end{cases} \quad (14)$$

Sufficient conditional maximum principle

Theorem 1: Let $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ with corresponding solutions $\hat{A}, \hat{X}, \hat{Y}, \hat{K}, \hat{\lambda}, \hat{p}, \hat{q}, \hat{r}$. Suppose that

- The functions $x \rightarrow h_i(x)$, $i = 1, 2$ and

$$(a, x, y, k, u) \rightarrow H(t, a, x, y, k, u, \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot))$$

are *concave*, for all $t \in [0, T]$

- $\hat{u}(t) \in \operatorname{argmax}_{v \in U} E[H(t, \hat{A}(t), \hat{X}(t), \hat{Y}(t), \hat{K}(t, \cdot), v, \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)) \mid \mathcal{E}_t]$

Then (under some growth conditions) $\hat{u}(t)$ is an optimal control i.e.

$$J(\hat{u}) = \sup_{u \in \mathcal{A}_{\mathcal{E}}} J(u).$$

Proof

Choose $u \in \mathcal{A}$ with corresponding solutions $A, X, Y, K, \lambda, p, q, r$.

We write

$$\hat{H}(t) = H(t, \hat{A}(t), \hat{X}(t), \hat{K}(t, \cdot), \hat{u}(t), \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot))$$

$$H(t) = H(t, A(t), X(t), Y(t), K(t, \cdot), u(t), \lambda(t), p(t), q(t), r(t)(t, \cdot))$$

and similarly with $\hat{f}(t), f(t), \dots$ etc.

$$J(\hat{u}) - J(u) = I_1 + I_2,$$

where

$$I_1 = E \left[\int_0^T \{\hat{f}(t) - f(t)\} dt \right]$$

and

$$I_2 = E[h_1(\hat{X}(0)) - h_1(X(0)) + h_2(\hat{A}(T)) - h_2(A(T))].$$

Using definition of $H = f + g\lambda + bp + \sigma q + \int_{\mathbb{R}_0} \gamma r \nu(dz)$ we have

$$\begin{aligned} I_1 = E \left[\int_0^T \{ \hat{H}(t) - H(t) - (\hat{g}(t) - g(t))\hat{\lambda}(t) \right. \\ \left. - (\hat{b}(t) - b(t))\hat{p}(t) - (\hat{\sigma}(t) - \sigma(t))\hat{q}(t) \right. \\ \left. - \int_{\mathbb{R}_0} (\hat{\gamma}(t, z) - \gamma(t, z))\hat{r}(t, z)\nu(dz) \} dt \right]. \end{aligned}$$

Since h_1 and h_2 are concave, we have

$$h_1(\hat{X}(0)) - h_1(X(0)) \geq (\hat{X}(0) - X(0))h_1'(\hat{X}(0)) = (\hat{X}(0) - X(0))\hat{\lambda}(0).$$

$$h_2(\hat{A}(T)) - h_2(A(T)) \geq (\hat{A}(T) - A(T))h_2'(\hat{A}(T)).$$

By the Itô formula and (9) and (14) we get

$$E[(\hat{X}(0) - X(0))\hat{\lambda}(0)] = E[(\hat{X}(T) - X(T))\hat{\lambda}(T)] \quad (1)$$

$$- E \left[\int_0^T (\hat{X}(t) - X(t)) d\hat{\lambda}(t) + \int_0^T \hat{\lambda}(t) d(\hat{X}(t) - X(t)) \right. \\ \left. + \int_0^T \frac{\partial \hat{H}}{\partial y}(t) (\hat{Y}(t) - Y(t)) dt + \int_0^T \int_{\mathbb{R}_0} \nabla_k \hat{H}(t, z) (\hat{K}(t, z) - K(t, z)) \nu(dz) dt \right] \quad (2)$$

$$(1) = E[(\hat{A}(T) - A(T))(\hat{p}(T) - h'_2(\hat{A}(T)))]$$

$$= E \left[\int_0^T (\hat{A}(t) - A(t)) d\hat{p}(t) + \int_0^T \hat{p}(t) d(\hat{A}(t) - A(t)) \right] \quad (3)$$

$$+ \int_0^T (\hat{\sigma}(t) - \sigma(t)) \hat{q}(t) dt + \int_0^T \int_{\mathbb{R}_0} (\hat{\gamma}(t, z) - \gamma(t, z)) \hat{r}(t, z) \nu(dz) dt$$

$$- E[(\hat{A}(T) - A(T))h'_2(\hat{A}(T))].$$

$$(3) = E \left[\int_0^T \left\{ (\hat{b}(t) - b(t)) \hat{p}(t) - (\hat{A}(t) - A(t)) \frac{\partial \hat{H}}{\partial a}(t) \right\} dt \right]$$

$$(2) = - E \left[\int_0^T \left\{ (\hat{X}(t) - X(t)) \frac{\partial \hat{H}}{\partial x}(t) - \hat{\lambda}(t) (\hat{g}(t) - g(t)) \right\} dt \right]$$



we get

$$\begin{aligned} & J(\hat{u}) - J(u) \\ & \geq E \left[\int_0^T \left\{ \hat{H}(t) - H(t) - (\hat{A}(t) - A(t)) \frac{\partial \hat{H}}{\partial a}(t) - (\hat{X}(t) - X(t)) \frac{\partial \hat{H}}{\partial x}(t) \right. \right. \\ & \quad \left. \left. - (\hat{Y}(t) - Y(t)) \frac{\partial \hat{H}}{\partial y}(t) - \int_{\mathbb{R}_0} \nabla_k \hat{H}(t, z) (\hat{K}(t, z) - K(t, z)) \nu(dz) \right\} dt \right] \\ & = E \left[\int_0^T E \left[\left\{ \hat{H}(t) - H(t) - (\hat{A}(t) - A(t)) \frac{\partial \hat{H}}{\partial a}(t) - (\hat{X}(t) - X(t)) \frac{\partial \hat{H}}{\partial x}(t) \right. \right. \right. \\ & \quad \left. \left. \left. - (\hat{Y}(t) - Y(t)) \frac{\partial \hat{H}}{\partial y}(t) - \int_{\mathbb{R}_0} \nabla_k \hat{H}(t, z) (\hat{K}(t, z) - K(t, z)) \nu(dz) \right\} \middle| \mathcal{E}_t \right] dt \right]. \end{aligned}$$



Since the function

$$(a, x, y, k, u) \rightarrow H(t, a, x, y, k, u, \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot))$$

is concave, we have

$$\begin{aligned} \hat{H}(t) - H(t) &\geq \frac{\partial \hat{H}}{\partial a}(t)(\hat{A}(t) - A(t)) + \frac{\partial \hat{H}}{\partial x}(t)(\hat{X}(t) - X(t)) \\ &\quad + \frac{\partial \hat{H}}{\partial y}(t)(\hat{Y}(t) - Y(t)) + \int_{\mathbb{R}_0} \nabla_k \hat{H}(t, z)(\hat{K}(t, z) - K(t, z))\nu(dz) \\ &\quad + \frac{\partial \hat{H}}{\partial u}(t)(\hat{u}(t) - u(t)). \end{aligned} \quad (15)$$

Since $\hat{u}(t) \in \operatorname{argmax} E[H(t, \hat{A}(t), \hat{X}(t), \hat{Y}(t), \hat{K}(t, \cdot), u, \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot))] | \mathcal{E}_t]$ we deduce that

$$\begin{aligned} \frac{d}{du} E[H(t, \hat{A}(t), \hat{X}(t), \hat{Y}(t), \hat{K}(t, \cdot), u, \\ \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot))] | \mathcal{E}_t]_{u=\hat{u}(t)} (\hat{u}(t) - u(t)) \geq 0 \end{aligned}$$

i.e.

$$E\left[\frac{\partial \hat{H}}{\partial u}(t)(\hat{u}(t) - u(t)) | \mathcal{E}_t\right] \geq 0.$$

A partial information equivalence principle for FBSDE's

Drawback with the previous result: the concavity assumption. Now we remove this assumption and assume the following instead:

(A1) $\forall s \in [0, T)$ and all bounded \mathcal{E}_s -measurable RV $\theta(\omega)$
the control

$$\beta_s(t) = \theta(\omega)\chi_{(s, T]}(t) ; t \in [0, T]$$

is in $\mathcal{A}_{\mathcal{E}}$.

(A2) $\forall u, \beta \in \mathcal{A}_{\mathcal{E}}$ where β is bounded, $\exists \delta > 0$ s.t. the
control

$$u(t) + y\beta(t) ; t \in [0, T]$$

belongs to $\mathcal{A}_{\mathcal{E}} \forall y \in (-\delta, \delta)$.

Partial information equivalence principle

Theorem 2: suppose $u \in \mathcal{A}_{\mathcal{E}}$ with corresponding solutions $A, X, Y, K, \lambda, p, q, r$. Then the following are equivalent:

(i) $\frac{d}{dy} J(u + y\beta) |_{y=0} = 0$ for all $\beta \in \mathcal{A}_{\mathcal{E}}$

(ii)
$$E \left[\frac{\partial}{\partial u} H(t, A(t), X(t), Y(t), K(t, \cdot), u, \lambda(t), p(t), q(t), r(t, \cdot)) \mid \mathcal{E}_t \right] = 0$$

Sketch of Proof

Define

$$\alpha(t) = \frac{d}{dy} A_{u+y\beta}(t) \Big|_{y=0}; \quad \xi(t) = \frac{d}{dy} X_{u+y\beta}(t) \Big|_{y=0}$$
$$\eta(t) = \frac{d}{dy} Y_{u+y\beta}(t) \Big|_{y=0}; \quad \zeta(t) = \frac{d}{dy} K_{u+y\beta}(t, z) \Big|_{y=0}$$

Note that

$$\alpha(0) = 0$$
$$\alpha(T) = \frac{1}{c} \frac{d}{dy} X_{u+y\beta}(T) \Big|_{y=0} = \frac{1}{c} \xi(T)$$

Assume that (i) holds. Then

$$\begin{aligned} 0 &= \frac{d}{dy} J(u + y\beta) |_{y=0} \\ &= E \left[\int_0^T \left\{ \frac{\partial f}{\partial a}(t)\alpha(t) + \frac{\partial f}{\partial x}\xi(t) + \frac{\partial f}{\partial y}(t)\eta(t) + \nabla_k f(t, z)\zeta(t, z) + \frac{\partial f}{\partial u}(t)\beta(t) \right. \right. \\ &\quad \left. \left. + h'_1(X(0))\xi(0) + h'_2(A(T))\alpha(T) \right] \end{aligned}$$



Using the Itô formula and after some computations, we get

$$E \left[\int_0^T \frac{\partial H}{\partial u}(t) \beta(t) dt \right] = 0 ; \beta \in \mathcal{A}_{\mathcal{E}} \text{ bounded .}$$

In particular, this holds for all $\beta \in \mathcal{A}_{\mathcal{E}}$ of the form

$$\beta(t) = \beta_s(t, \omega) = \theta(\omega) \chi_{[s, T]}(t) ; t \in [0, T] \quad (16)$$

for a fixed $s \in [0, T]$ where $\theta(\omega)$ is a bounded \mathcal{E}_s -measurable RV.



This gives

$$E \left[\int_s^T \frac{\partial H}{\partial u}(t) \theta dt \right] = 0.$$

Differentiating with respect to s we arrive at

$$E \left[\frac{\partial H}{\partial u}(s) \theta \right] = 0.$$

Since this holds for all bounded \mathcal{E}_s -measurable random variables θ , we conclude that

$$E \left[\frac{\partial H}{\partial u}(s) \mid \mathcal{E}_s \right] = 0.$$

This proves that (i) \Rightarrow (ii).

Conversely, since every bounded $\beta \in \mathcal{A}_{\mathcal{E}}$ can be approximated by linear combinations of controls β_s of the form (16), we can prove that (ii) \Rightarrow (i) by reversing the above argument.



A Malliavin calculus approach

- ▶ Replace the adjoint processes p, q, r given by BSDEs by $\tilde{p}, \tilde{q}, \tilde{r}$ given directly in terms of the parameters and state of the system.

$$\begin{cases} dA(t) = b(t, A(t), u(t), \omega)dt + \sigma(t, A(t), u(t), \omega)dB(t) \\ \quad + \int_{\mathbb{R}} \gamma(t, A(t), u(t), z, \omega)\tilde{N}(dt, dz); \quad t \in [0, T] \\ A(0) = a \in \mathbb{R} \end{cases}$$

$$\begin{cases} dX(t) = -g(t, A(t), X(t), Y(t), u(t), \omega)dt + Y(t)dB(t) \\ \quad + \int_{\mathbb{R}} K(t, z)\tilde{N}(dt, dz); \quad t \in [0, T] \\ X(T) = cA(T), \quad c \in \mathbb{R} \setminus \{0\} \end{cases}$$

$$J(u) = E \left[\int_0^T f(t, A(t), X(t), Y(t), K(t, \cdot), u(t), \omega) dt + h_1(X(0)) + h_2(A(T), \omega) \right];$$

$b(t, a, u, \omega)$ is \mathcal{F}_t -measurable for each constant a, u , and similarly with σ, γ, g and f .



Modified Hamiltonian

$$\tilde{H}(t, a, x, y, k, u, \lambda, \omega) = f + \lambda g + b\tilde{p}(t) + \sigma\tilde{q}(t) + \int_{\mathbb{R}_0} \gamma\tilde{r}(t, z)\nu(dz),$$

where

$$\tilde{p}(t) = K(t) + \int_t^T \frac{\partial H_0}{\partial a}(s)G(t, s)ds$$

$$\tilde{q}(t) = D_t\tilde{p}(t)$$

$$\tilde{r}(t, z) = D_{t,z}\tilde{p}(t)$$

$D_t F$: Malliavin derivative wrt $B(\cdot)$ (at t) of a rv F .

$D_{t,z} F$: Malliavin derivative wrt $\tilde{N}(\cdot, \cdot)$ (at t, z) of F .

(Di Nunno, Øksendal and Proske (Springer 2009))

$$\tilde{p}(t) = K(t) + \int_t^T \frac{\partial H_0}{\partial a}(s) G(t, s) ds$$

with

$$K(t) = h_2'(A(T)) + c\tilde{\lambda}(T) + \int_t^T \frac{\partial f}{\partial a}(s) ds$$

$$H_0(s, a, x, u) = \tilde{\lambda}(s)g + K(s)b + D_s K(s)\sigma + \int_{\mathbb{R}_0} D_{s,z} K(s) \gamma \nu(dz)$$

$$G(t, s) = \exp \left(\int_t^s \left\{ \frac{\partial b}{\partial a}(r) - \frac{1}{2} \left(\frac{\partial \sigma}{\partial a}(r) \right)^2 \right\} dr + \int_t^s \frac{\partial \sigma}{\partial a}(r) dB(r) \right. \\ \left. + \int_t^s \int_{\mathbb{R}_0} \ln(1 + \frac{\partial \gamma}{\partial a}(r, z)) \tilde{N}(dr, dz) + \int_0^t \int_{\mathbb{R}_0} [\ln(1 + \frac{\partial \gamma}{\partial a}) - \frac{\partial \gamma}{\partial a}] \nu(dz) dr ; s > t \right)$$

$\tilde{\lambda}(t)$ given by the same (forward) equation as for λ but H replaced by \tilde{H} :

$$\begin{aligned}d\tilde{\lambda}(t) &= \frac{\partial \tilde{H}}{\partial x}(t, A(t), X(t), Y(t), K(t, \cdot), u(t), \tilde{\lambda}(t))dt \\ &\quad + \frac{\partial \tilde{H}}{\partial y}(\cdot)dB(t) + \int_{\mathbb{R}_0} \nabla_k \tilde{H}(\cdot) \tilde{N}(dr, dz); \quad t \in [0, T] \\ \tilde{\lambda}(0) &= h'_1(X(0)).\end{aligned}\tag{17}$$



Main result

Theorem 3: Let $u \in \mathcal{A}_\varepsilon$ with corresponding solutions $A(t), X(t), Y(t), K(t, z)$ and $\tilde{\lambda}(t)$. Assume that $h'_2(A(T)), \lambda(T), \frac{\partial f}{\partial a}(t)$ and $\frac{\partial H_0}{\partial a}(s)G(t, s)$ are Malliavin differentiable to all $s > t$ and satisfy the conditions for the use of duality formulae when necessary. Then the following are equivalent:

1. $\frac{d}{dy}J(u + y\beta) |_{y=0} = 0$ for all bounded $\beta \in \mathcal{A}_\varepsilon$
2. $E \left[\frac{d}{du} \tilde{H}(t, A(t), X(t), Y(t), K(t), u, \tilde{\lambda}(t))_{u=u(t)} \mid \mathcal{E}_t \right] = 0$
for a.a. $(t, \omega) \in [0, T] \times \Omega$.

Duality formulae for Malliavin derivatives

$$E \left[F \int_0^T \varphi(s) dB(s) \right] = E \left[\int_0^T \varphi(s) D_s F ds \right]$$

$$E \left[F \int_0^T \int_{\mathbb{R}_0} \psi(s, z) \tilde{N}(ds, dz) \right] = E \left[\int_0^T \int_{\mathbb{R}_0} \psi(s, z) D_{s,z} F \nu(dz) ds \right],$$

valid for all Malliavin differentiable F and \mathcal{F}_t -predictable processes φ and ψ such that the integrals on the right converge absolutely.

Application to risk minimizing portfolios

- Wealth process $A(t) = A_u(t)$:

$$\begin{cases} dA(t) = u(t) \left[\alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}_0} \theta(t, z)\tilde{N}(dt, dz) \right] \\ A(0) = a > 0 \end{cases} \quad (18)$$

where α, β and θ are given predictable processes.

$u(t) = \pi(t)A(t^-)$: amount invested in the risky asset at time t .

- Corresponding BSDE for (X, Y, K) :

$$\begin{cases} dX(t) = -g(t, X(t), \omega)dt + Y(t)dB(t) + \int_{\mathbb{R}_0} K(t, z)\tilde{N}(dt, dz) \\ X(T) = -A_u(T) \end{cases} \quad (19)$$

g is a function s.t. $(t, \omega) \rightarrow g(t, x, \omega)$ is \mathcal{F}_t -predictable $\forall x$.

- Performance functional:

$$J(u) = X_u(0)$$



Modified Hamiltonian:

$$\begin{aligned}\tilde{H}(t, a, x, y, k, u, \lambda, \omega) \\ = \lambda g(t, x) + u\alpha(t)\tilde{p}(t) + u\beta(t)\tilde{q}(t) + \int_{\mathbb{R}_0} u\theta(t, z)\tilde{r}(t, z)\nu(dz),\end{aligned}$$

where

$$\tilde{p}(t) = K(t) + \int_t^T \frac{\partial H_0}{\partial a}(s)G(t, s)ds = -\tilde{\lambda}(T)$$

$$\tilde{q}(t) = -D_t\tilde{\lambda}(T)$$

$$\tilde{r}(t, z) = -D_{t,z}\tilde{\lambda}(T)$$

and $\tilde{\lambda}(t)$ is given by:

$$d\tilde{\lambda}(t) = \tilde{\lambda}(t)g'(t, \hat{X}(t))dt; \quad \tilde{\lambda}(0) = 1,$$

Hence

$$\tilde{\lambda}(t) = \exp\left(\int_0^t g'(s, \hat{X}(s))ds\right); \quad 0 \leq t \leq T.$$

Condition for an optimal control $\hat{u}(t)$:

$$E \left[\alpha(t) \tilde{\lambda}(T) + \beta(t) D_t \tilde{\lambda}(T) + \int_{\mathbb{R}_0} \theta(t, z) D_{t,z} \tilde{\lambda}(T) \nu(dz) \mid \mathcal{E}_t \right] = 0.$$

If $\mathcal{E}_t = \mathcal{F}_t$ we can solve this *Malliavin-differential* eq in the unknown rv $\tilde{\lambda}(T)$:

$$\begin{aligned} \tilde{\lambda}(T) = E[\tilde{\lambda}(T)] \exp & \left(\int_0^T \sigma(s) dB(s) - \frac{1}{2} \int_0^T \sigma^2(s) ds \right. \\ & + \int_0^T \int_{\mathbb{R}_0} \ln(1 + \gamma(s, z)) \tilde{N}(ds, dz) \\ & \left. + \int_0^T \int_{\mathbb{R}_0} \{ \ln(1 + \gamma(s, z)) - \gamma(s, z) \} \nu(dz) ds \right) \end{aligned}$$

for some (any) \mathcal{F}_t -predictable processes $\sigma(t)$ and $\gamma(t, z)$ such that

$$\alpha(t) + \beta(t) \sigma(t) + \int_{\mathbb{R}_0} \theta(t, z) \gamma(t, z) \nu(dz) = 0 \text{ for a.a. } t, \omega. \quad (20)$$

Condition (20) says that the measure Q defined by

$$dQ(\omega) = \frac{\tilde{\lambda}(T)}{E[\tilde{\lambda}(T)]} dP(\omega) \text{ on } \mathcal{F}_T \text{ is an ELMM for the process } A(t).$$


A special case

Suppose $g(t, x, \omega) = c_0(t) + c(t)x$; $\mathcal{E}_t = \mathcal{F}_t$. Then the solution X_u of

$$dX(t) = -g(t, X(t), \omega)dt + Y(t)dB(t) + \int_{\mathbb{R}_0} K(t, z)\tilde{N}(dt, dz); \quad X(T) = -A_u(T)$$

satisfies

$$X_u(0) = -E \left[\lambda_u(T)A_u(T) - \int_0^T \lambda_u(t)c_0(t)dt \right] \quad \text{with}$$

$$\lambda_u(t) = \exp \left(\int_0^t g'(s, X_u(s))ds \right) = \exp \left(\int_0^t c(s)ds \right).$$

Suppose an optimal control \hat{u} exists. Then the minimal risk is

$$\begin{aligned} X_{\hat{u}}(0) &= -E[\lambda_{\hat{u}}(T)]E_Q[A_{\hat{u}}(T)] + E\left[\int_0^T \lambda_{\hat{u}}(t)c_0(t)dt\right] \\ &= -aE[\lambda_{\hat{u}}(T)] + E\left[\int_0^T \lambda_{\hat{u}}(t)c_0(t)dr\right] \\ &= -aE\left[\exp\left(\int_0^T c(s)ds\right)\right] + \int_0^T E\left[c_0(t)\exp\left(\int_0^t c(s)ds\right)\right] dt, \end{aligned}$$

and the optimal control is $\hat{u} = 0$.

